Bachelor Thesis

General Relativity: An alternative derivation of the Kruskal-Schwarzschild solution

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Introduction

The Einstein equations

The field equations of General Relativity (A. Einstein, 1915):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad c = 1$$

We consider the following special case:

- Vacuum spacetime ($T_{\mu\nu} = 0$)
- Zero cosmological constant ($\Lambda = 0$)

Then we have the reduced field equations

$$R_{\mu\nu} = 0$$

The Schwarzschild solution

A solution of the vacuum field equations in case of spherical symmetry (K. Schwarzschild, 1916):

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad G = c = 1$$

The metric has two singularities:

- $r = 0$
- $r = r_s = 2m, \quad r_s = \text{Schwarzschild radius}$

Consider the curvature invariant $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = 48m^2/r^6$, then:

- $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ diverges for $r \to 0$!
- $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ is finite for $r = r_s$!

Coordinate singularities

The singularity at $r = r_s$ is a coordinate singularity! What does this mean?

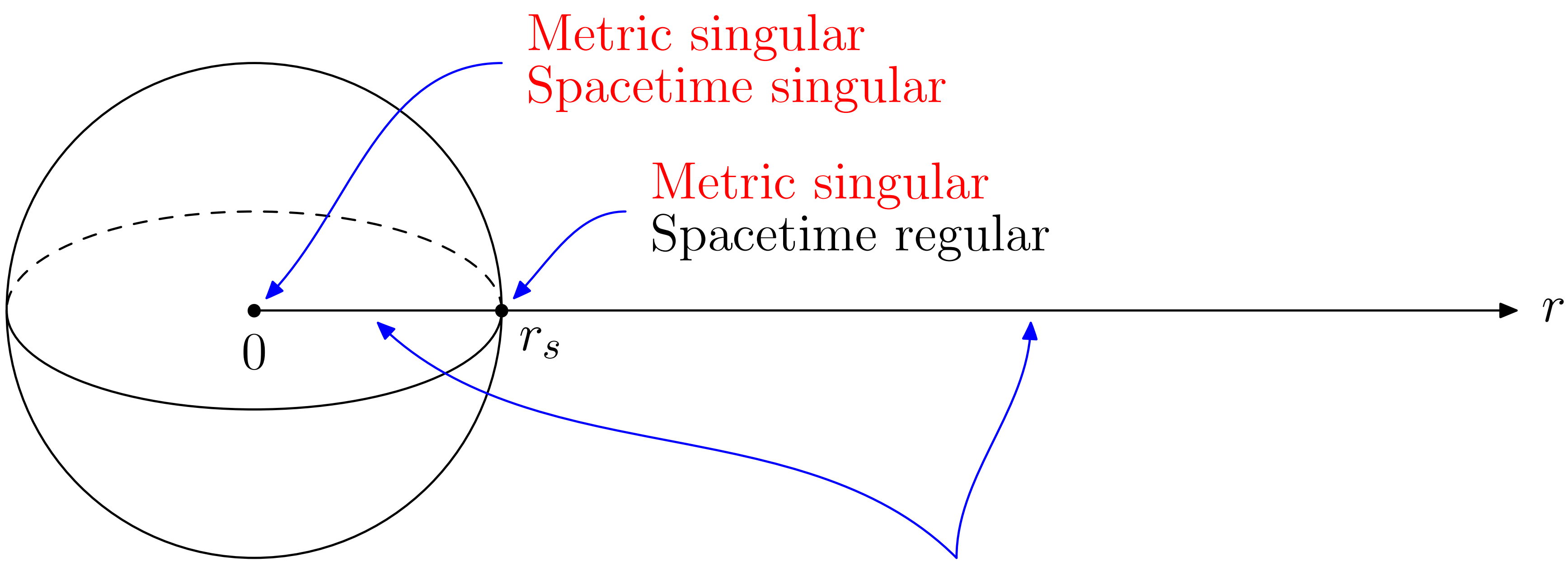
Consider the Minkowski spacetime in spherical coordinates, given by the metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

The metric is singular at $r = 0$, but we know that flat spacetime has no true spacetime singularities!

$\Rightarrow$ The coordinates are badly behaved at coordinate singularities!

The Schwarzschild black hole



$$\overbrace{ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2}$$

The Kruskal transformation

Is it possible to cover the interior and the exterior region of the Schwarzschild black hole with one coordinate patch?

Yes! Consider the coordinate transformation (M. D. Kruskal, 1960):

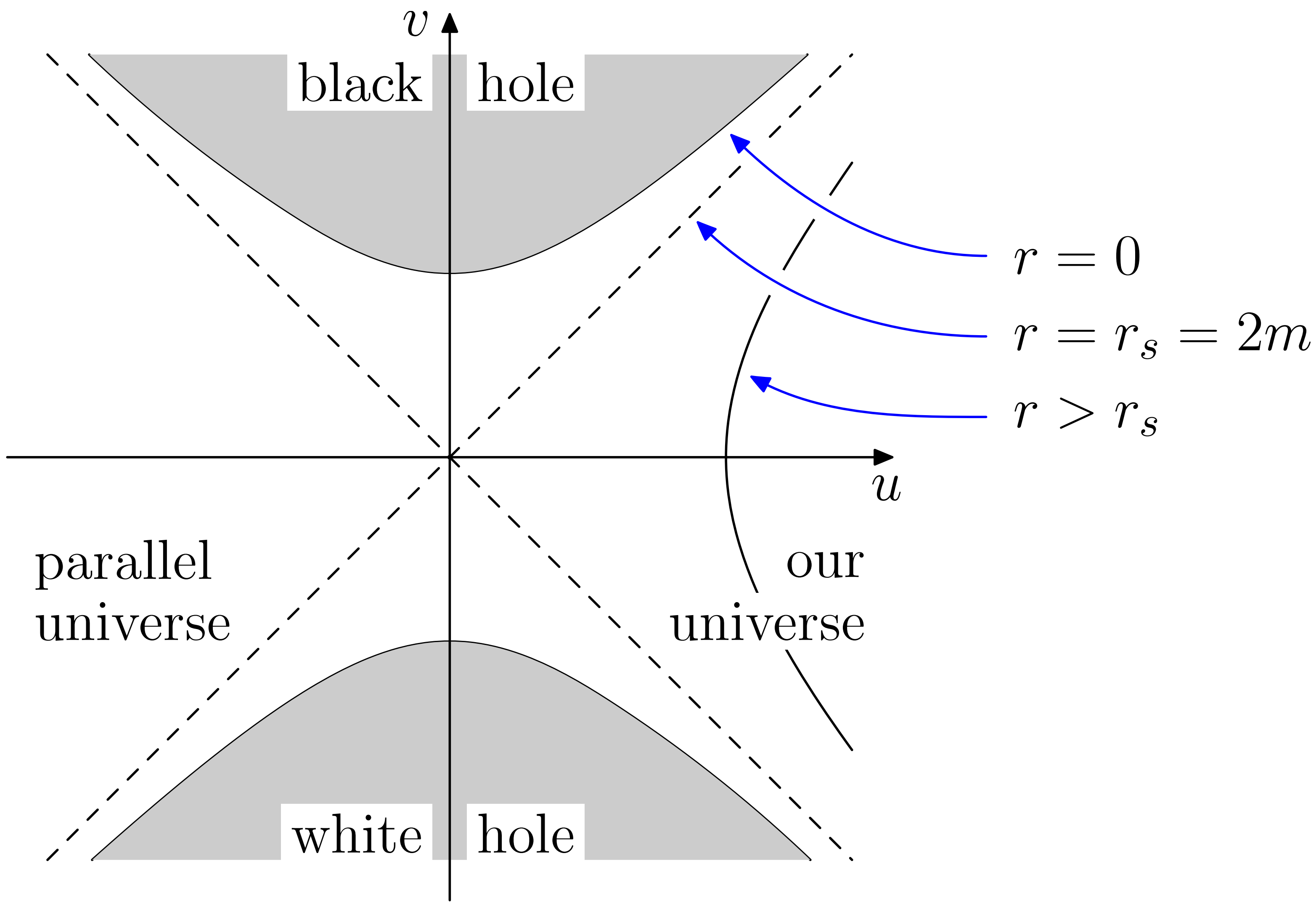
$$\begin{pmatrix} u \\ v \end{pmatrix} = \sqrt{\frac{r}{2m} - 1}\, e^{r/4m} \begin{pmatrix} \cosh\left(\frac{t}{4m}\right) \\ \sinh\left(\frac{t}{4m}\right) \end{pmatrix}$$

The metric now becomes

$$\begin{aligned} ds^2 &= \frac{32m^3}{r} e^{-r/2m}(-dv^2 + du^2) + r^2 d\Omega^2 \\ u^2 - v^2 &= \left(\frac{r}{2m} - 1\right) e^{r/2m} \end{aligned}$$

and it is regular at $r = r_s = 2m$!

The Kruskal-Schwarzschild solution



What about uniqueness?

What spherically symmetric maximal solutions do exist?

- Kruskal-Schwarzschild spacetime ($m > 0$)
- Minkowki spacetime ($m = 0$)
- A solution with $m < 0$

Are there more solutions?

- In case of analytic solutions: No! (Birkhoff theorem, theorem of analytic continuation)
- In case of $\mathcal{C}^2$-solutions: Probably no! But there doesn't seem to exist a fully satisfactory proof concerning $\mathcal{C}^2$-uniqueness!

Formulating the goals

- Rederive the spherically symmetric maximal solutions in a new way, based on an idea by M. Heusler.
- Try not to make any arbitrary choices during the derivation (such as staticity, analyticity etc.)
- If possible, prove the uniqueness of the solutions on $\mathcal{C}^2$-level.
- Be mathematically precise!
- Try to get some insights about the origin of the 4-fold structure of the Kruskal-Schwarzschild spacetime.

Other works in this area

- T.M. Kalotas and L. Rizzo (1980): An alternative path to the Kruskal extension of the Schwarzschild metric
  - Makes arbitrary assumptions
- M.O. Katanaev, T. Klösch and W. Kummer (1999). Global properties of warped solutions in General Relativity
  - Does carry out a similar program, but in a wider context (more general spacetimes)
  - The maximal solutions are not constructed explicitly
  - Still no satisfactory uniqueness proof

We will follow the second of these works for some time!

Deriving the Kruskal-Schwarzschild solution

Spherical symmetry

By spherical symmetry, the metric must take the following form:

$$ds^2 = \underbrace{\tilde{g}_{ij}(x^0, x^1)dx^i dx^j}_{\tilde{g}} + F^2(x^0, x^1)d\Omega^2, \quad (i, j) \in \{0, 1\}$$

Or equivalently:

$$ds^2 = \begin{pmatrix} \tilde{g}_{00}(x^0, x^1) & \tilde{g}_{01}(x^0, x^1) & 0 & 0 \\ \tilde{g}_{10}(x^0, x^1) & \tilde{g}_{11}(x^0, x^1) & 0 & 0 \\ 0 & 0 & F^2(x^0, x^1) & 0 \\ 0 & 0 & 0 & F^2(x^0, x^1)\sin^2\theta \end{pmatrix}$$

$\Rightarrow$ The spacetime is a warped product of the two pseudo-riemmanian surfaces $\tilde{g}$ and S^2!

Specializing the field equations (1)

- Calculate the Ricci tensor for the simplified metric
- Plug it into the vacuum field equations $R_{\mu\nu} = 0$

$\Rightarrow$ The field equations break up into the following system:

$$\begin{aligned}
&(1) & \tilde{\Delta}(F^2) &= 2 \\
&(2) & \left(\tilde{\nabla}_i \tilde{\nabla}_j - \frac{1}{2}\tilde{g}_{ij}\tilde{\Delta}\right) F &= 0 \\
&(3) & \frac{1}{F}\tilde{\Delta} F &= \frac{\tilde{R}}{2}
\end{aligned}$$

It can be shown, that (1) $\wedge$ (2) $\Rightarrow$ (3)! So we can forget about (3)!

Conformal flatness

Every pseudo-riemannian surface is conformally flat $\Rightarrow$ there exist co-ordinates in which the metric takes the form:

$$\tilde{g} = \pm\omega^2(\sigma,\tau)(-d\tau^2 + d\sigma^2)$$

Now we introduce the null coordinates u and v:

$$\sigma = \frac{u+v}{2}, \quad \tau = \frac{u-v}{2}$$

The metric takes the form

$$\begin{aligned} \tilde{g} &= \pm\omega^2(u,v)(dudv + dvdu) \\ \Rightarrow ds^2 &= \pm\omega^2(u,v)(dudv + dvdu) + F^2(u,v)d\Omega^2 \end{aligned}$$

Specializing the field equations (2)

Using the even more simplified metric, we can again specialize the field equations:

$$\pm \frac{2}{\omega^2}(\partial_u F \partial_v F + F \partial_u \partial_v F) = 1 \tag{4}$$

$$\partial_u^2 F - \frac{2}{\omega}\partial_u \omega \partial_u F = 0 \tag{5}$$

$$\partial_v^2 F - \frac{2}{\omega}\partial_v \omega \partial_v F = 0 \tag{6}$$

These are three partial differential equations for two unknown functions $F(u, v)$ and $\omega(u, v)$.

The first integration

Consider the equations (5) and (6):

$$\partial_u^2 F - \frac{2}{\omega}\partial_u\omega\partial_u F = 0, \qquad \partial_v^2 F - \frac{2}{\omega}\partial_v\omega\partial_v F = 0$$

We want to divide these equations by $\partial_u F$ resp. $\partial_v F$ in order to perform one integration, but $\partial_u F$ and $\partial_v F$ might not be nonzero everywhere!

$\Rightarrow$ We need to remove all points with $\partial_u F = 0$ and $\partial_v F = 0$ from the domain of integration! The local integration yields

$$|\partial_u F| - e^{-C_1(v)}\omega^2 = 0$$
$$|\partial_v F| - e^{-C_2(u)}\omega^2 = 0$$

where $C_1(v)$ and $C_2(u)$ are arbitrary $\mathcal{C}^2$-functions.

Fixing the gauge freedom

Again consider the (integrated) equations (5) and (6):

$$|\partial_u F| - e^{-C_1(v)}\omega^2 = 0, \qquad |\partial_v F| - e^{-C_2(u)}\omega^2 = 0$$

The functions $C_1(v)$ and $C_2(u)$ represent a local gauge freedom! In order to advance we need to choose a gauge.

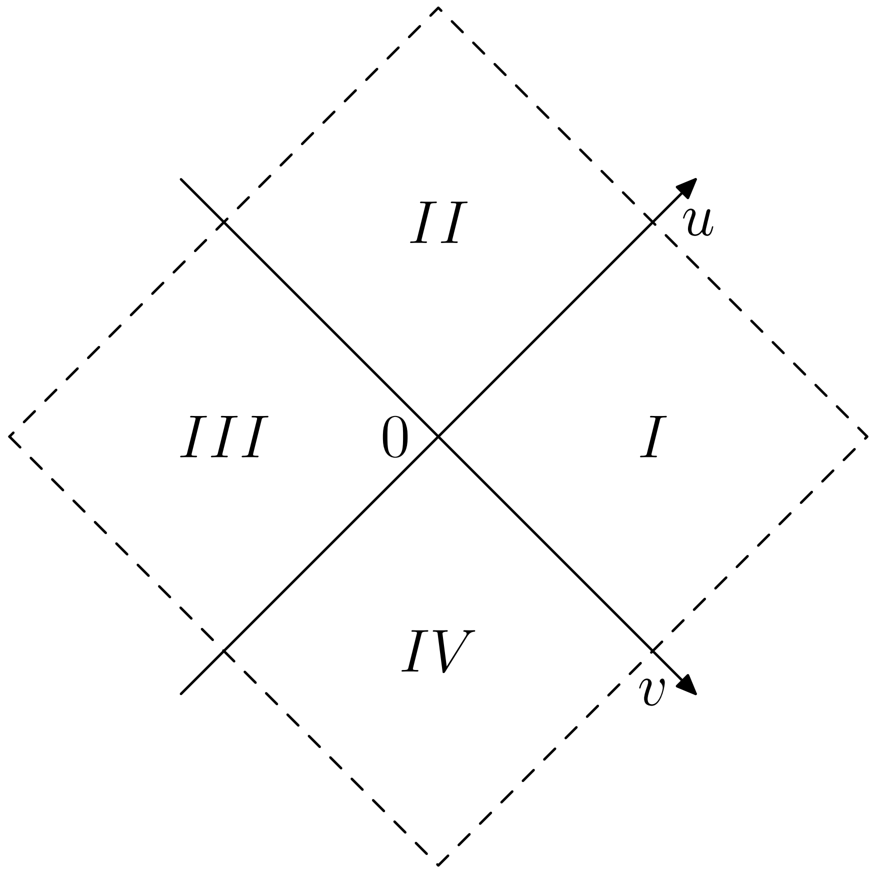
- Katanaev, Klösch, Kummer: $e^{-C_1(v)} = e^{-C_2(u)} = \mathrm{const}$
- Our choice:

$$\begin{matrix} e^{-C_1(v)} = \eta|v| \\ e^{-C_2(u)} = \eta|u| \end{matrix}\ , \quad \eta > 0$$

This choice only makes sense if $u \neq 0$ and $v \neq 0$! So we need to remove all the points with $u = 0$ and $v = 0$ from the domain of integration.

The four domains of integration

We have twice removed points from the whole (u,v)-plane. But in our gauge, the removed points are the same! So the (u,v)-plane splits up into four local domains of integration:



The field equations in our gauge

After having fixed the gauge freedom, the field equations and the metric read:

$$\begin{aligned} \pm\frac{2}{\omega^2}(\partial_u F \partial_v F + F\partial_u \partial_v F) &= 1 \\ \partial_u F \pm \eta v \omega^2 &= 0 \\ \partial_v F \pm \eta u \omega^2 &= 0 \end{aligned}$$

$$ds^2 = \pm\omega^2(u,v)(dudv + dvdu) + F^2(u,v)d\Omega^2$$

Note, that three of the $\pm$-signs are independent from eachother: in fact there are 8 different cases to distinguish (this is handled more properly in the Thesis).

The second integration

After some calculations, another integration can be performed, yielding

$$
\begin{aligned}
\frac{dF}{d\xi} &= \pm\left(1+\frac{C}{2F}\right), \quad \xi = \xi(u,v) \\
\omega^2 &= \pm\frac{1}{2\eta^2 uv}\left(1+\frac{C}{2F}\right) \\
ds^2 &= \pm\omega^2(dudv + dvdu) + F^2 d\Omega^2
\end{aligned}
$$

where ξ is an auxiliary coordinate. Note, that we have determined one of the two unknowns in terms of the other: $\omega^2(u,v,F)$!

C is an integration constant, which does not represent a gauge freedom! It will rather be associated to the mass of the system. $C = 0$ instantly leads to the Minkowski spacetime!

The final integration

The remaining differential equation

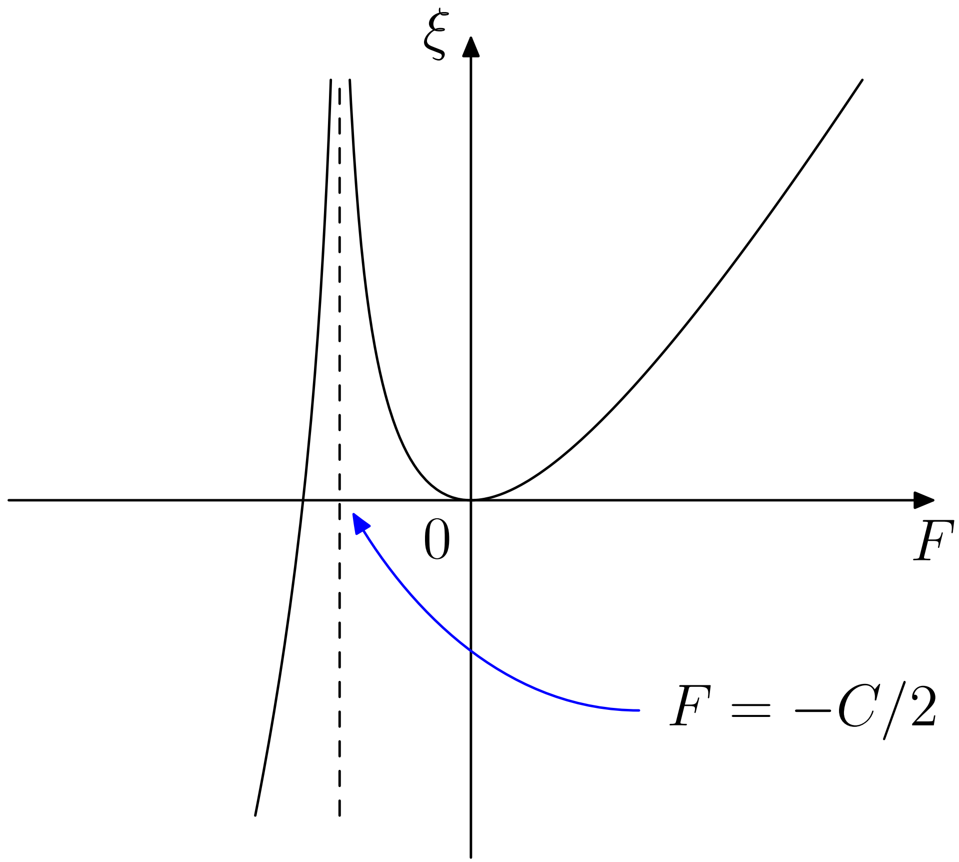
$$\frac{dF}{d\xi} = \pm\left(1 + \frac{C}{2F}\right)$$

can be integrated, leading to

$$\xi(F) = \pm\left(F - \frac{C}{2}\log|2F + C| + C'\right)$$

This equation can not be solved explicitly for $F(\xi)$!

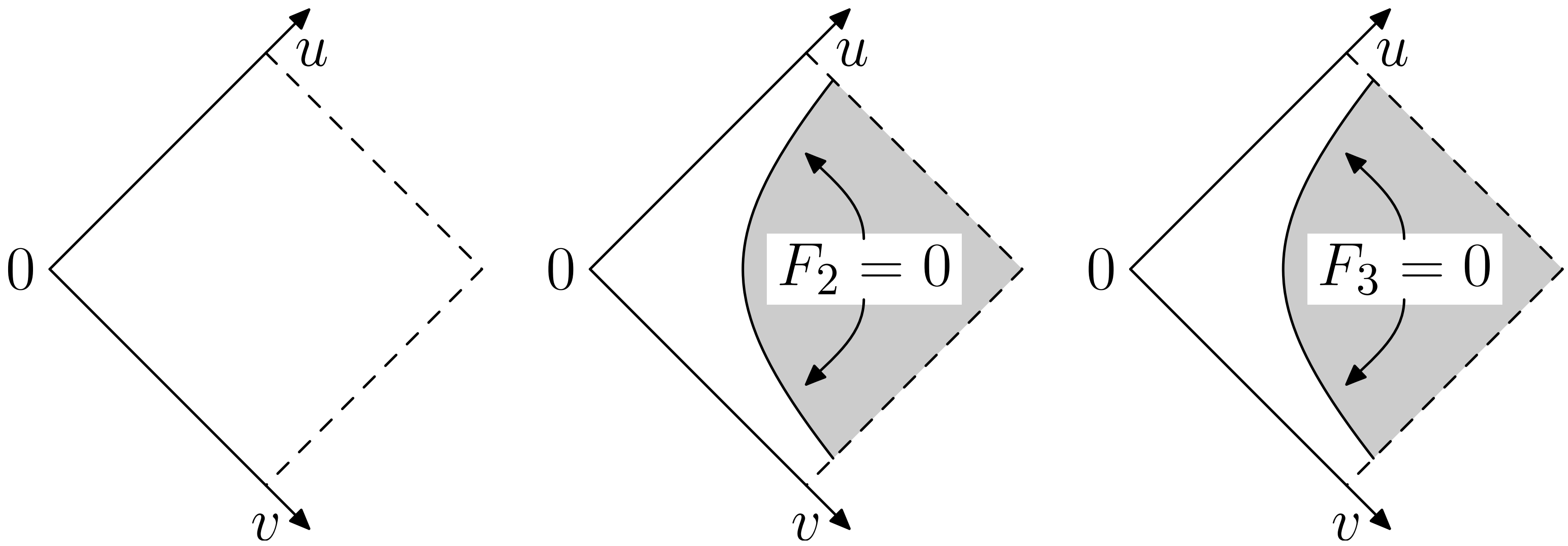
The function $\xi(F)$



By local inversion we get three functions $F_{1-3}(\xi)$, which all solve the field equations.

Domain and image of $F_{1-3}(\xi(u,v))$

$F_1 \in \left[-\infty, -\frac{C}{2}\right]$ $F_2 \in \left[-\frac{C}{2}, 0\right]$ $F_3 \in [0, \infty]$



Finalizing the local integration

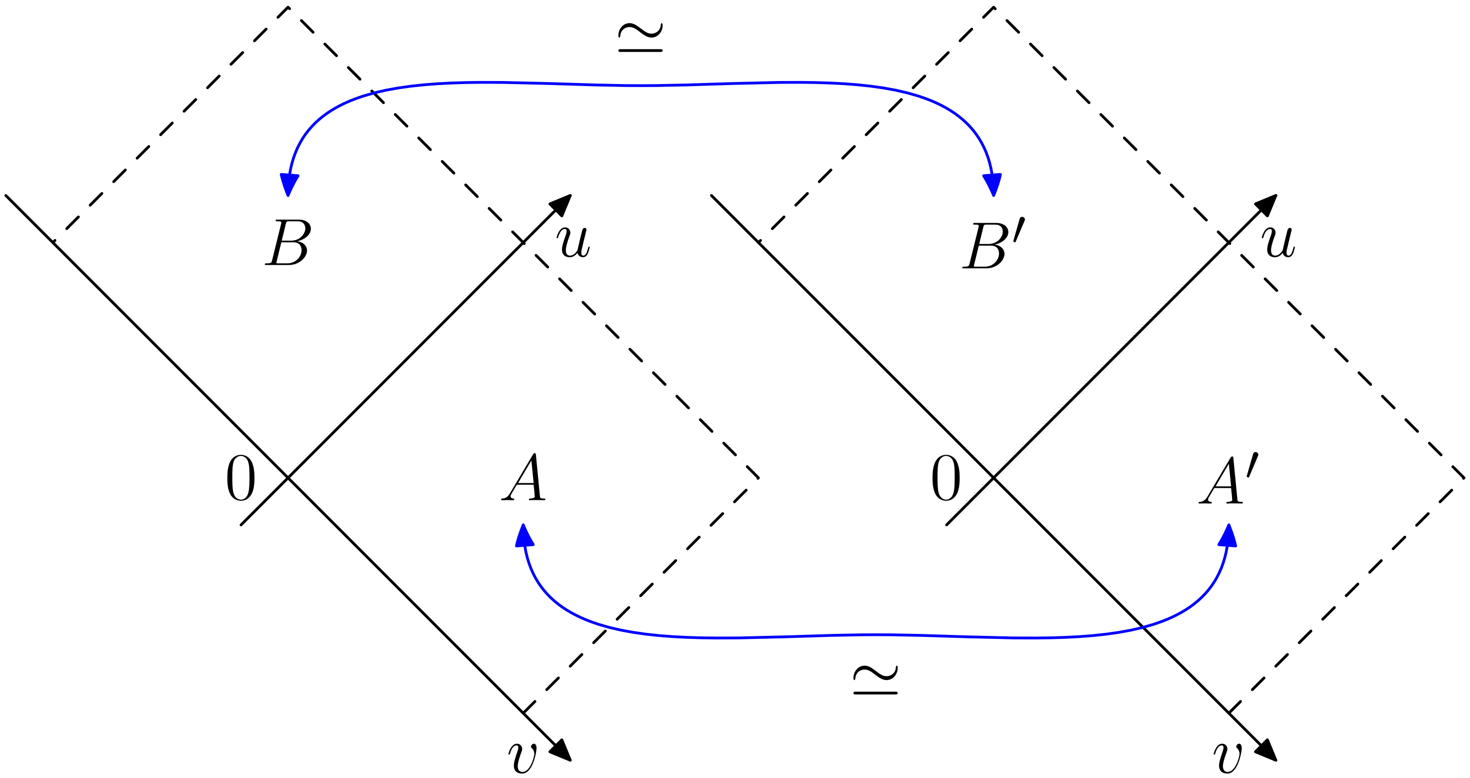
After some calculations we get the following result:

$$
\begin{aligned}
2uv(F) &= -e^{-\frac{2F}{C}}\left(1+\frac{2F}{C}\right) \\
ds^2 &= -\frac{C^3}{2F}e^{\frac{2F}{C}}(dudv+dvdu)+F^2 d\Omega^2
\end{aligned}
$$

These formulas describe three local solutions corresponding to the three functions F_{1-3} discussed previously (modulo local and global gauge freedoms).

They are the only inequivalent local solutions!

Local vs. global gauge freedom



If $A \simeq A'$ and $B \simeq B'$, are the global solutions AB and $A'B'$ in general equivalent, too? In the following we will assume the answer to be negative.

Gluing local solutions together

We construct the global solutions by gluing local solutions together.

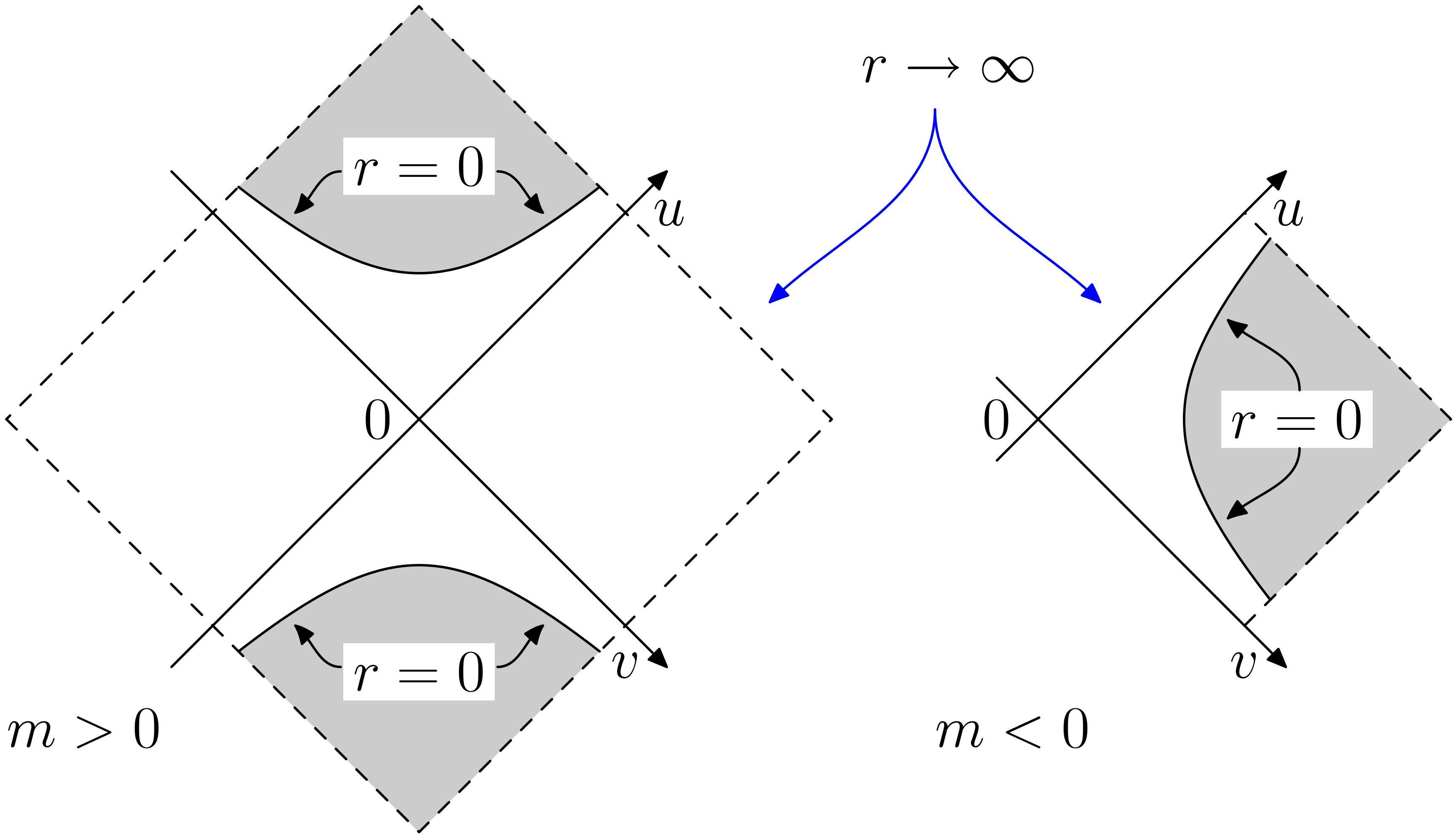
In the Thesis the following result was proven:

- In the chosen local gauge there exist only three maximal $\mathcal{C}^2$-solutions:
  - The Minkowski spacetime
  - The Kruskal-Schwarzschild spacetime
  - A solution with negative mass

To yield a general proof of $\mathcal{C}^2$-uniqueness, we need the gauge-independence of this result! To achieve this, we need a 'yes' as the answer to the question asked on the previous slide!

This issue could not be resolved in this work.

The maximal solutions with $m \neq 0$



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