Bachelor Thesis

General Relativity: Derivation of all maximal spherically symmetric vacuum solutions

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Abstract

The vacuum Einstein equations with zero cosmological constant are integrated in the case of spherical symmetry and all maximal solutions are constructed explicitely using suitable coordinate systems, without using analytic continuation. This pedagogical paper covers all the steps in detail, starting with the abstract field equations and ending with the analytic expressions of the maximal solutions.

1 Introduction

Solving the vacuum Einstein equations in case of spherical symmetry is a classic problem in General Relativity. Only a few months after the publication of the theory of General Relativity in 1916, Karl Schwarzschild derived the metric describing the spacetime around a center of mass, which is meanwhile known as the *Schwarzschild metric*:

$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Here M is the total mass of the system and G is the gravitational constant.

This metric is singular at both $r = 0$ and $r = r_s = 2GM/c^2$ (r_s is called the *Schwarzschild radius*). A number of publications [1] showed, that these two singularities are of a very different character: while the spacetime is truly singular at $r = 0$, it is regular at $r = r_s$, which reveals the second singularity as a coordinate singularity, separating the exterior and the interior region of the *Schwarzschild black hole*. In terms of differential geometry this means, that the Schwarzschild coordinates fail to cover the entire manifold.

It was not until 1960, when M. D. Kruskal [1] presented a simple coordinate transformation, which removes the coordinate singularity and yields a metric that describes the exterior and the interior of the Schwarzschild black hole with the same coordinate chart. Furthermore the maximal analytic extension of the Schwarzschild manifold, today known as *Kruskal-Schwarzschild manifold*, was shown to have an even more complex topology, not only consisting of a black hole, but also of a white hole and even a parallel universe isometric to the original Schwarzschild manifold.

When deriving the Kruskal-Schwarzschild manifold (described by the *Kruskal metric*) starting with the original Schwarzschild manifold, one needs to apply the theorem of analytic continuation. Not many attempts have been made so far to derive the Kruskal metric alternatively without using analytic continuation (see [2] for example).

The question of uniqueness of the Kruskal-Schwarzschild spacetime seems to be simple at a first glance. This is mainly due to the well-known *Birkhoff Theorem*, which states, that the exterior region of every spherically symmetric solution of the vacuum Einstein equations with zero cosmological constant is uniquely described by the Schwarzschild metric. The uniqueness of the Kruskal-Schwarzschild spacetime then follows from the theorem of analytic continuation, provided we restrict ourselves to analytic maximal solutions.

It is fairly natural to assume, that dropping the analyticity requirement should not yield additional solutions, but a rigorous and transparent proof to our knowledge has not been published yet. In 1999 Katanaev et al. [3]

released a comprehensive analysis of warped spacetimes, which also contains the special case of spherical symmetry. The global solutions are not constructed explicitely using global coordinates, instead a generic algorithm is given how to glue local charts together. Applying the algorithm to the spherically symmetric case they derive the well-known spherically symmetric global solutions without using analytic continuation. The uniqueness of the derived global solutions is claimed, but the explanations concerning the uniqueness have been kept short, so we think, that a more elaborate approach might be of interest.

In this present work we derive all maximal spherically symmetric solutions of the vacuum Einstein equations with zero cosmological constant. The local integration is performed in a similar way as done in [3]. We carry out all steps in detail so that this work should be useful for students attempting to learn how to integrate the Einstein equations.

2 Warped products of manifolds

Before we start with the actual integration process we discuss warped products of manifolds, because we will need some of the results derived here later.

Let $\tilde{M}$ and $\hat{M}$ be pseudo-riemannian manifolds, described by the metrics

$$\begin{aligned} ds^2 &= \tilde{g}_{ij}dx^i dx^j, \quad i,j \in \{0,\ldots,m-1\} \\ ds^2 &= \hat{g}_{AB}dx^A dx^B, \quad A,B \in \{m,\ldots,m+n-1\} \end{aligned}$$

with $n = \dim(\tilde{M})$ and $m = \dim(\hat{M})$. A *warped product* of these manifolds is a $m+n$-dimensional manifold described by a metric of the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \tilde{g}_{ij}dx^i dx^j + F^2(x^k)\hat{g}_{AB}dx^A dx^B \tag{1}$$

$\mu,\nu \in \{0,\ldots,m+n-1\},\ i,j,k \in \{0,\ldots,m-1\},\ A,B \in \{m,\ldots,m+n-1\}$

We want to express the Ricci tensor of the warped product manifold in terms of the Ricci tensors of $\tilde{M}$ and $\hat{M}$. The calculation of the Christoffel symbols yields

$$\Gamma^i{}_{jk} = \tilde{\Gamma}^i{}_{jk}, \quad \Gamma^A{}_{BC} = \hat{\Gamma}^A{}_{BC}, \quad \Gamma^A{}_{ij} = 0$$

$$\Gamma^A{}_{Bi} = \frac{1}{F}\delta^A_B \partial_i F, \quad \Gamma^i{}_{AB} = -F(\partial_j F)\tilde{g}^{ij}\hat{g}_{AB}, \quad \Gamma^i{}_{Aj} = 0$$

For the Ricci tensor components we get

$$\begin{aligned} R_{ij} &= \tilde{R}_{ij} - \frac{1}{F}\tilde{\nabla}_i\tilde{\nabla}_j F \dim(\hat{M}) && (2) \\ R_{AB} &= \hat{R}_{AB} - \hat{g}_{AB}\Big(F\tilde{\Delta}F + (\dim(\hat{M})-1)(\tilde{\nabla}_i F\tilde{\nabla}^i F)\Big) && (3) \\ R_{Ai} &= 0 && (4) \end{aligned}$$

Here, $\tilde{\nabla}$ is the covariant derivative operator on $\tilde{M}$ and $\tilde{\Delta} = \tilde{\nabla}_i \tilde{\nabla}^i = \tilde{g}^{ij} \tilde{\nabla}_i \tilde{\nabla}_j$ is the covariant Laplace operator on $\tilde{M}$.

The Ricci scalar of the 2-sphere

As an application of warped products we consider the 2-sphere with the standard metric

$$ds^2 = g_{AB} dx^A dx^B = d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{5}$$

Comparing this metric with (1) we see that the 2-sphere is a warped product of two one-dimensional manifolds. Carrying out the procedure outlined above, we calculate the Christoffel symbols

$$\Gamma^\theta{}_{\theta\theta} = 0, \quad \Gamma^\phi{}_{\phi\phi} = 0, \quad \Gamma^\phi{}_{\theta\theta} = 0$$

$$\Gamma^\phi{}_{\phi\theta} = \cot\theta, \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\theta{}_{\phi\theta} = 0$$

and the Ricci tensor components

$$R_{\theta\theta} = 1, \quad R_{\phi\phi} = \sin^2\theta, \quad R_{\theta\phi} = 0$$

Finally we find the Ricci scalar of the 2-sphere:

$$R_{S^2} = 2 \tag{6}$$

Now the Einstein tensor vanishes on every pseudo-riemannian surface (see Proposition 1 in Appendix A.1): $G_{AB} = 0 \Rightarrow R_{AB} = \frac{1}{2} g_{AB} R$. Therefore the 2-sphere has the following property:

$$R_{AB,S^2} = g_{AB,S^2} \tag{7}$$

Manifolds with $R_{AB} = \text{const} \cdot g_{AB}$ are called *Einstein manifolds*. The 2-sphere is a two-dimensional example of an Einstein manifold.

3 Local integration of the field equations

Now we can start the actual integration process. First we examine the property of spherical symmetry in order to simplify the metric, then the field equations are specialized to the simplified metric. Finally the resulting differential equations are integrated locally. Throughout the whole paper we use the signature convention `(-+++)`.

3.1 Spherical symmetry

The following definition is quoted from Wald's book on general relativity [4]:

> A spacetime is said to be *spherically symmetric* if its isometry group contains a subgroup isomorphic to the group SO(3) and the orbits of this subgroup (i.e., the collection of points resulting from the action of the subgroup on a given point) are two-dimensional spheres. The SO(3) isometries may then be interpreted physically as rotations, and thus a spherically symmetric spacetime is one whose metric remains invariant under rotations.[1]

We wish to simplify the metric in its most general form

$$ds^2 = g_{\mu\nu}(x^0, x^1, x^2, x^3)dx^\mu dx^\nu, \quad (\mu, \nu) \in \{0 \dots 3\} \tag{8}$$

with the help of the definition above. First we choose spherical coordinates by setting $x^2 = \theta, x^3 = \phi$. Now the definition says, that the metric is invariant under rotations, in particular also under the coordinate transformation $x'^0 = x^0, x'^1 = x^1, \theta' = -\theta, \phi' = \phi$. Transforming the metric tensor to the primed coordinates with

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

and setting $g'_{\mu\nu} = g_{\mu\nu}$ one immediately obtains the condition

$$g_{\mu\theta} = 0 \ (\mu \neq \theta)$$

and a similar argument leads to $g_{\mu\phi} = 0$ $(\mu \neq \phi)$. So the metric takes the block diagonal form

$$ds^2 = g_{ij}dx^i dx^j + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2, \quad (i, j) \in \{0, 1\}$$

By spherical symmetry, the restriction of the metric to the surfaces of constant angles does not depend on the angles themselves: $g_{ij} = g_{ij}(x^0, x^1)$. On the other hand, the restriction of the metric to the surfaces of constant x^0 and x^1 has to be a multiple of the standard metric of the 2-sphere:

$$g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 = \alpha(x^0, x^1)d\Omega^2$$

Our signature convention implies $\alpha > 0$, so that the metric takes the form

$$ds^2 = \tilde{g}_{ij}(x^0, x^1)dx^i dx^j + F^2(x^0, x^1)d\Omega^2, \quad (i, j) \in \{0, 1\} \tag{9}$$

which is a warped product of two pseudo-riemannian surfaces, one of them being the 2-sphere.

[1] We also require the surfaces of the spheres to be spacelike.

3.2 Reduction of the field equations

Our next task is to specialize the vacuum field equations

$$R_{\mu\nu} = 0 \tag{10}$$

to the spherically symmetric case, which gives us the reduced field equations. Since we have simplified the metric into a warped product, we can use the results derived in section 2 to write down new expressions for the Ricci tensor. We proceed in a similar way as done in [3].

In the following we denote the (i, j)-block of the metric (9) with $\tilde{g}$:

$$\tilde{g} = \tilde{g}_{ij}(x^0, x^1)dx^i dx^j$$

Since $\tilde{g}$ represents a pseudo-riemannian surface, we have $\tilde{G}_{ij} = 0$ (see Appendix A.1, Proposition 1) and thus

$$\tilde{R}_{ij} = \frac{1}{2}\tilde{g}_{ij}\tilde{R} \tag{11}$$

Combining the formulas (2)-(4),(7) and (11) we get

$$\begin{aligned} R_{ij} &= \frac{1}{2}\tilde{g}_{ij}\tilde{R} - \frac{2}{F}\tilde{\nabla}_i\tilde{\nabla}_j F && (12)\\ R_{AB} &= \hat{g}_{AB}(1 - F\tilde{\Delta}F - \tilde{\nabla}_i F\tilde{\nabla}^i F) && (13)\\ R_{Ai} &= 0 && (14) \end{aligned}$$

Here A and B are the indices running over the angles.

Now we write down the field equations using the formulas (12)-(14). The field equation associated to R_{AB} is

$$F\tilde{\Delta}F + \tilde{\nabla}_i F\tilde{\nabla}^i F = 1 \tag{15}$$

On the other hand, the field equation associated to R_{ij} reads

$$\frac{1}{2}\tilde{g}_{ij}\tilde{R} = \frac{2}{F}\tilde{\nabla}_i\tilde{\nabla}_j F \tag{16}$$

Multiplication of (16) with $\tilde{g}^{ij}$ and contracting all indices afterwards leads to

$$\frac{1}{F}\tilde{\Delta}F = \frac{1}{2}\tilde{R} \tag{17}$$

which, inserted back into (16), gives us the traceless part of (16):

$$\left(\tilde{\nabla}_i\tilde{\nabla}_j - \frac{1}{2}\tilde{g}_{ij}\tilde{\Delta}\right)F = 0 \tag{18}$$

(15),(17) and (18) are the reduced field equations. They are not independent: if a metric satisfies (15) and (18), then it also satisfies (17). See Appendix A.2, Lemma 2 for the proof of this assertion. Now the remaining reduced field equations can be summarized:

$$F\tilde{\Delta}F + \tilde{\nabla}_i F\tilde{\nabla}^i F = 1 \tag{19}$$

$$\left(\tilde{\nabla}_i\tilde{\nabla}_j - \frac{1}{2}\tilde{g}_{ij}\tilde{\Delta}\right) F = 0 \tag{20}$$

Formula (19) can be simplified. A short calculation leads to the relation $\tilde{\Delta}(F^2) = 2(F\tilde{\Delta}F + \tilde{\nabla}_i F\tilde{\nabla}^i F)$, so that the formula (19) reduces to $\tilde{\Delta}(F^2) = 2$. We do not use this expression in the subsequent work.

Conformal flatness

Now we focus back to the metric (9). We can take advantage of an interesting fact concerning pseudo-riemannian surfaces: all of them are *conformally flat*, which means that there exist local coordinates in which the metric takes the form $ds^2 = \alpha(\sigma, \tau)(\pm d\tau^2 \pm d\sigma^2)$. In Appendix A.1 Proposition 2 we give a proof for the case, when the surface has signature (1,1), which is the case applicable to $\tilde{g}$. So we can assume $\tilde{g}$ to take the form

$$\tilde{g} = \alpha(\sigma, \tau)(-d\tau^2 + d\sigma^2)$$

Now we define the *null coordinates* u and v by

$$\sigma = \frac{u+v}{2}, \quad \tau = \frac{u-v}{2}$$

so that $\tilde{g}$ now reads

$$\tilde{g} = \alpha(u, v)(dudv + dvdu)$$

The terminology *null coordinates* comes from the fact, that the coordinate axes lie on the light cone of the observer at the origin. To simplify calculations done later we set

$$\alpha(u, v) = s\omega^2(u, v), \quad s \in \{-1, 1\}$$

where we have introduced the binary switch s representing a distinction of cases. Finally the metric (9) takes the form

$$ds^2 = s\omega^2(u, v)(dudv + dvdu) + F^2(u, v)d\Omega^2 \tag{21}$$

With the Christoffel symbols associated to $\tilde{g} = s\omega^2(u,v)(dudv + dvdu)$

$$\tilde{\Gamma}^u{}_{uu} = \frac{2\partial_u\omega}{\omega}, \quad \tilde{\Gamma}^v{}_{vv} = \frac{2\partial_v\omega}{\omega}, \quad \tilde{\Gamma}^v{}_{uu} = 0$$

$$\tilde{\Gamma}^v{}_{vu} = 0, \quad \tilde{\Gamma}^u{}_{vv} = 0, \quad \tilde{\Gamma}^u{}_{vu} = 0$$

we can transform the reduced field equations (19) and (20) to the newly defined coordinates u and v:

$$s\frac{2}{\omega^2}(\partial_u F \partial_v F + F\partial_u\partial_v F) = 1 \tag{22}$$

$$\partial_u^2 F - \frac{2}{\omega}\partial_u\omega\partial_u F = 0 \tag{23}$$

$$\partial_v^2 F - \frac{2}{\omega}\partial_v\omega\partial_v F = 0 \tag{24}$$

Note, how the number of equations have been effectively reduced: the original field equations (10) represented 10 equations, taking the symmetry of the metric tensor into account. Then they have been reduced to four equations during the first reduction process (equations (19) and (20)) and the transformation to the (u,v)-coordinates resulted into another reduction to three equations.

3.3 Local integration, part 1

Our next task is to integrate the reduced field equations (22)-(24). They represent a system of partial differential equations for the unknown functions $F(u,v)$ and $\omega(u,v)$. Fortunately the partial derivatives have been separated to some extent: (23) and (24) do only contain partial derivatives with respect to one variable. This is not just coincidence, but a consequence of the choice of null coordinates.

Examining the equations (23) and (24), we see that it would be advantageous if we could divide them by $\partial_u F$ resp. $\partial_v F$. In general these partial derivatives do not have to be nonvanishing on the entire manifold. Thus we need to remove all the points p with $\partial_u F(p) = 0$ and $\partial_v F(p) = 0$ from the domain of integration and to take care of these removed points later.

We place the origin of our coordinate system at a point p with $\partial_u F(p) \neq 0$ and $\partial_v F(p) \neq 0$ and perform the division explained above. The resulting equations can be integrated, yielding

$$s\frac{2}{\omega^2}(\partial_u F \partial_v F + F\partial_u\partial_v F) = 1 \tag{25}$$

$$|\partial_u F| - e^{-C_1(v)}\omega^2 = 0 \tag{26}$$

$$|\partial_v F| - e^{-C_2(u)}\omega^2 = 0 \tag{27}$$

with arbitrary $\mathcal{C}^1$-functions $C_1(v)$ and $C_2(u)$. These functions represent a gauge freedom. To see this, consider any map $(u, v) \to (u', v')$ obtained by solving the following ordinary differential equations:

$$\frac{du'}{du} = e^{C_2(u)}, \quad \frac{dv'}{dv} = e^{C_1(v)}$$

Since the Jacobi determinant of the map is nonvanishing everywhere, the map is a diffeomorphism onto its image and thus a coordinate transformation. Transforming the equations (25)-(27) and the metric tensor we get

$$s\frac{2}{\omega'^2}(\partial_{u'} F \partial_{v'} F + F \partial_{u'} \partial_{v'} F) = 1 \tag{28}$$

$$|\partial_{u'} F| - \omega'^2 = 0 \tag{29}$$

$$|\partial_{v'} F| - \omega'^2 = 0 \tag{30}$$

where ω' is defined by

$$s\omega^2(dudv + dvdu) = s\omega'^2(du'dv' + dv'du')$$

Comparing the transformed equations with the original ones we find that for every pair of $\mathcal{C}^1$-functions $(C_1(v), C_2(u))$ there exists a coordinate transformation which transforms (28)-(30) into (25)-(27), and so the solutions with different $C_1(v)$ and $C_2(u)$ are also connected by coordinate transformations.

Fixing the gauge freedom

The choice of a specific gauge is an important step during the integration process, because different choices lead to different technical problems during the subsequent integration and they also lead to different insights concerning the structure of the equations and solutions. The most simple gauge is to set $e^{-C_1(v)} = e^{-C_2(u)} = \text{const}$, which was the gauge chosen by the authors of [3]. The coordinates corresponding to this gauge turn out to be the well-known *tortoise coordinates*. We choose another gauge, which will turn out to correspond to the equally well-known *kruskal coordinates*.

To use the new gauge we need to modify the domain of integration by removing all points with $u = 0$ and $v = 0$. Now we can write down the gauge definition:

$$e^{-C_1(v)} = \eta|v|, \quad e^{-C_2(u)} = \eta|u|, \quad \eta = \text{const} > 0, \quad u \neq 0, \quad v \neq 0 \tag{31}$$

We have actually not fixed the gauge completely, instead we leave a part of the gauge freedom open by leaving the constant η unspecified. Now, the equations (26) and (27) become

$$|\partial_u F| = \eta|v|\omega^2, \quad |\partial_v F| = \eta|u|\omega^2 \tag{32}$$

(s_1, s_2, s_3, s_4)	I $u > 0$ $v > 0$	II $u > 0$ $v < 0$	III $u < 0$ $v < 0$	IV $u < 0$ $v > 0$
$\partial_u F > 0, \partial_v F > 0$	(1, 1, 1, 1)	(-1, 1, 1,-1)	(-1,-1,-1,-1)	(1,-1,-1, 1)
$\partial_u F > 0, \partial_v F < 0$	(1,-1, 1, 1)	(-1,-1, 1,-1)	(-1, 1,-1,-1)	(1, 1,-1, 1)
$\partial_u F < 0, \partial_v F < 0$	(-1,-1, 1, 1)	(1,-1, 1,-1)	(1, 1,-1,-1)	(-1, 1,-1, 1)
$\partial_u F < 0, \partial_v F > 0$	(-1, 1, 1, 1)	(1, 1, 1,-1)	(1,-1,-1,-1)	(-1,-1,-1, 1)

Table 1: Definition of the binary switches s_1-s_4 to deal with case distinctions

Up to this point we have reduced the domain of integration by removing all points with $u = 0$, $v = 0$, $\partial_u F = 0$ and $\partial_v F = 0$. But we have

$$u = 0 \Leftrightarrow \partial_v F = 0, \quad v = 0 \Leftrightarrow \partial_u F = 0 \tag{33}$$

as a consequence of the equations (32), therefore all points removed have $u = 0$ or $v = 0$. We conclude, that the (u, v)-plane splits up into four local domains of integration, which we denote with the roman letters I-IV. The conditions (33) additionally tell us, that the signs of $\partial_u F$ and $\partial_v F$ are well defined in each domain.

At this point it is important to recognize that the gauge fixing was done separately in each of the four domains of integration! In particular, the constant η can take on different values in each domain. While this fact is irrelevant for the local integration, it becomes important when attempting to construct the global solutions by gluing local solutions together.

The modulus signs in the equations (32) lead to a distinction of 16 cases. Since we will integrate all cases simultaneously we define four binary switches $s_1, s_2, s_3, s_4 \in \{-1, 1\}$ (see Table 1). The meaning of the switches s_3 and s_4 is easily recognized; they just describe the domain of integration. For example $(s_3, s_4) = (1, 1)$ marks the domain I.[2] With these switches we can dispose of all modulus signs and write the field equations as follows:

$$s\frac{2}{\omega^2}(\partial_u F \partial_v F + F \partial_u \partial_v F) = 1 \tag{34}$$

$$\partial_u F - s_1 \eta v \omega^2 = 0 \tag{35}$$

$$\partial_v F - s_2 \eta u \omega^2 = 0 \tag{36}$$

Since all entities in these equations are nonzero we can safely divide (35) by

[2]Now we could write $\eta = \eta(s_3, s_4)$ but we do not use this notation to keep the formulas compact.

$s = 1$	I/III	II/IV
r^*	spacelike	timelike
t^*	timelike	spacelike

$s = -1$	I/III	II/IV
r^*	timelike	spacelike
t^*	spacelike	timelike

Table 2: The timelike/spacelike character of the (r^*, t^*)-coordinates

(36) giving us the relations

$$\partial_u F = s_1 s_2 \frac{v}{u} \partial_v F, \quad \partial_v F = s_1 s_2 \frac{u}{v} \partial_u F \tag{37}$$

Introducing new coordinates

Equation (34) is still too complicated to be integrated directly. Therefore we need to define new coordinates (r^*, t^*) by

$$r^* = \frac{1}{2\eta} \log(s_3 s_4 2uv), \quad t^* = \frac{1}{2\eta} \log(s_3 s_4 \frac{u}{v}) \tag{38}$$

The inverse transformation reads

$$u = s_3 \frac{1}{\sqrt{2}} e^{\eta(r^*+t^*)}, \quad v = s_4 \frac{1}{\sqrt{2}} e^{\eta(r^*-t^*)} \tag{39}$$

With the useful relation $s_3 s_4 2uv = e^{2\eta r^*}$ we can transform the metric (21):

$$ds^2 = s s_3 s_4 \omega^2 \eta^2 e^{2\eta r^*} (-dt^{*2} + dr^{*2}) + F^2 d\Omega^2 \tag{40}$$

This formula reveals the character of r^* resp. t^* (see Table 2). Using (37) we can calculate $\partial_{r^*} F$ and $\partial_{t^*} F$ as follows:

$$\partial_{r^*} F = \partial_u F \eta u + \partial_v F \eta v = \partial_u F \eta u + s_1 s_2 \partial_u F \eta u \tag{41}$$
$$\partial_{t^*} F = \partial_u F \eta u - \partial_v F \eta v = \partial_u F \eta u - s_1 s_2 \partial_u F \eta u \tag{42}$$

These two expressions reveal an important property of the unknown function F: it never depends on both coordinates at the same time. In particular we have

$$\begin{aligned} s_1 s_2 = 1 &\Rightarrow \partial_{t^*} F = 0 \\ s_1 s_2 = -1 &\Rightarrow \partial_{r^*} F = 0 \end{aligned}$$

We are led to define the new coordinate ξ by

$$\xi = \begin{cases} r^* & \text{if } s_1 s_2 = 1 \\ t^* & \text{if } s_1 s_2 = -1 \end{cases}$$

which allows us to derive the following simple expression from the equations (41) and (42):

$$\frac{dF}{d\xi} = 2\eta u \partial_u F \tag{43}$$

Now we are ready to transform the field equations (34)-(36). With the aid of the formulas (43) and (37) we find the expressions

$$\partial_u F = \frac{1}{2\eta u}\frac{dF}{d\xi}, \quad \partial_v F = s_1 s_2 \frac{1}{2\eta v}\frac{dF}{d\xi}, \quad \partial_u \partial_v F = s_1 s_2 \frac{1}{4\eta^2 uv}\frac{d^2F}{d\xi^2}$$

The field equations (35) and (36) both transform into

$$\omega^2 = s_1 \frac{1}{2\eta^2 uv}\frac{dF}{d\xi} \tag{44}$$

while the transformed field equation (34) now reads

$$s s_1 s_2 \frac{2}{\omega^2}\frac{1}{4\eta^2 uv}\left(\left(\frac{dF}{d\xi}\right)^2 + F\frac{d^2F}{d\xi^2}\right) = 1 \tag{45}$$

Inserting (44) into (45) we get

$$\left(\frac{dF}{d\xi}\right)^2 + F\frac{d^2F}{d\xi^2} = s s_2 \frac{dF}{d\xi} \quad \Rightarrow \quad \frac{d^2F^2}{d\xi^2} = s s_2 2\frac{dF}{d\xi}$$

which is an ordinary differential equation. Its integration leads to

$$\frac{dF^2}{d\xi} = s s_2 2F + \tilde{C} \quad \Rightarrow \quad \frac{dF}{d\xi} = s s_2 + \frac{\tilde{C}}{2F}$$

with the integration constant $\tilde{C}$. We want to emphasize, that at this point $\tilde{C}$ can take different values in the different cases selected by s, s_1, s_2, s_3 and s_4! It would be more precise to write $\tilde{C}(s, s_1, s_2, s_3, s_4)$, but we do not use this notation to prevent the formulas to blow up in size. For convenience purposes we define

$$C = s s_2 \tilde{C}$$

The field equations and the metric now read

$$\begin{aligned}
\frac{dF}{d\xi} &= s s_2 \left(1 + \frac{C}{2F}\right) && (46)\\
\omega^2 &= s s_1 s_2 \frac{1}{2\eta^2 uv}\left(1 + \frac{C}{2F}\right) && (47)\\
ds^2 &= s\omega^2(dudv + dvdu) + F^2 d\Omega^2 && (48)\\
&= s_1 s_2 \left(1 + \frac{C}{2F}\right)(-dt^{*2} + dr^{*2}) + F^2 d\Omega^2 && (49)
\end{aligned}$$

Equation (49) follows directly from (40) and (47), again using the relation $s_3 s_4 2uv = e^{2\eta r^*}$.

3.4 The flat solution

In this section we discuss the case $C = 0$. The field equation (46) becomes $dF/d\xi = ss_2 \Rightarrow F = ss_2\xi + C'$ with the integration constant C'. Again we suppress the dependency of C' on the binary switches s,s_1-s_4. We notice, that $F(\xi)$ is a diffeomorphism from $]-\infty, \infty[$ to $]-\infty, \infty[$ and hence a coordinate transformation for all C'. It follows, that C' is a gauge freedom, which we fix with $C' = 0 \Rightarrow F = ss_2\xi$. The metric now reads

$$ds^2 = s_1 s_2(-dt^{*2} + dr^{*2}) + \xi^2 d\Omega^2$$

With the definition of new coordinates (r, t)

$$(r,t) = \begin{cases} (r^*, t^*) & \text{if } s_1 s_2 = 1 \\ (t^*, r^*) & \text{if } s_1 s_2 = -1 \end{cases}$$

the metric takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \qquad (r \in\,]-\infty, \infty[)$$

In these coordinates, the metric is singular at $r = 0$, so we actually get two equivalent local solutions, one for $r < 0$ and the other one for $r > 0$. Transforming the metric to cartesian coordinates we get in both cases

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad (x, y, z) \neq (0, 0, 0)$$

In these coordinates the metric remains finite and nonzero for $r \to 0$, so that we can extend the solution by adding all the points with $(x, y, z) = (0, 0, 0)$. The final result reads

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

and describes the flat spacetime, also known as *Minkowski spacetime.* A spacetime is called *maximal*, if every inextendible geodesic in any direction either runs infinitely with respect to the affine parameter or runs into a true spacetime singularity for a finite value of this parameter. The Minkowski spacetime is easily seen to be maximal, so that it is the unique maximal spherically symmetric solution of the field equations in the case $C = 0$.

3.5 Exploring symmetries

Before we continue with the case $C \neq 0$, we analyse the symmetries of the field equations in the gauge (31):

$$s\frac{2}{\omega^2}(\partial_u F \partial_v F + F \partial_u \partial_v F) = 1 \tag{50}$$

$$|\partial_u F| = \eta |v| \omega^2 \tag{51}$$

$$|\partial_v F| = \eta |u| \omega^2 \tag{52}$$

We want to find those transformations which leave each equation invariant. Consider an arbitrary coordinate transformation $(u, v) \to (u', v')$. The invariance condition is then stated as:

$$\begin{aligned} |\partial_u F| = \eta |v| \omega^2 &\Leftrightarrow |\partial_{u'} F| = \eta |v'| \omega'^2 \\ |\partial_v F| = \eta |u| \omega^2 &\Leftrightarrow |\partial_{v'} F| = \eta |u'| \omega'^2 \\ s\omega^2(dudv + dvdu) + F^2 d\Omega^2 &= s'\omega'^2(du'dv' + dv'du') + F^2 d\Omega^2 \end{aligned}$$

With the transformation of the metric

$$\begin{aligned} s'\omega'^2 &= s\omega^2(\partial_u u' \partial_v v' + \partial_v u' \partial_u v') \\ \Rightarrow \omega'^2 &= \omega^2 |\partial_u u' \partial_v v' + \partial_v u' \partial_u v'| \end{aligned}$$

we can rewrite equations (51) and (52) in terms of the new coordinates

$$|\partial_{u'} F \partial_u u' + \partial_{v'} F \partial_u v'| - \eta |v| \omega'^2 |\partial_u u' \partial_v v' + \partial_v u' \partial_u v'| = 0 \tag{53}$$

$$|\partial_{u'} F \partial_v u' + \partial_{v'} F \partial_v v'| - \eta |u| \omega'^2 |\partial_u u' \partial_v v' + \partial_v u' \partial_u v'| = 0 \tag{54}$$

The invariance condition lead us to impose the following restriction on the coordinate transformation:

$$\partial_u v' = 0, \quad \partial_v u' = 0$$

The equations (53) and (54) now read

$$|\partial_{u'} F| - \eta |v| \omega'^2 \left| \frac{dv'}{dv} \right| = 0, \quad |\partial_{v'} F| - \eta |u| \omega'^2 \left| \frac{du'}{du} \right| = 0$$

and the invariance condition leads to the following ordinary differential equations for the unknown functions $u'(u)$ and $v'(v)$:

$$\left| \frac{du'}{du} \right| = \left| \frac{u'}{u} \right|, \quad \left| \frac{dv'}{dv} \right| = \left| \frac{v'}{v} \right|$$

(u', v')	I $(s_3, s_4) =$ $(\ \ 1, \ \ 1)$	II $(s_3, s_4) =$ $(\ \ 1, -1)$	III $(s_3, s_4) =$ $(-1, -1)$	IV $(s_3, s_4) =$ $(-1, \ \ 1)$
$(s_1, s_2) = (\ \ 1, \ \ 1)$	(u, v)	$(u, -v)$	$(-u, -v)$	$(-u, v)$
$(s_1, s_2) = (\ \ 1, -1)$	$(\tilde{u}, v)$	$(\tilde{u}, -v)$	$(-\tilde{u}, -v)$	$(-\tilde{u}, v)$
$(s_1, s_2) = (-1, -1)$	$(\tilde{u}, \tilde{v})$	$(\tilde{u}, -\tilde{v})$	$(-\tilde{u}, -\tilde{v})$	$(-\tilde{u}, \tilde{v})$
$(s_1, s_2) = (-1, \ \ 1)$	$(u, \tilde{v})$	$(u, -\tilde{v})$	$(-u, -\tilde{v})$	$(-u, \tilde{v})$

Table 3: Transformations connecting the 16 solutions described by s_1-s_4

They have the following solutions:

$$u' = \pm\alpha_1 u, \quad u' = \pm\alpha_2 u^{-1}, \quad v' = \pm\alpha_3 v, \quad v' = \pm\alpha_4 v^{-1}, \quad \alpha_i > 0 \qquad (55)$$

A short calculation verifies, that these transformations also leave the third field equation (50) invariant. (55) describes eight one-parameter families of maps and now we choose a representative of each family by setting the scaling factors to some values that will prove to be convenient later:

$$u' = \pm u, \quad u' = \pm\tilde{u}, \quad v' = \pm v, \quad v' = \pm\tilde{v}$$

Here we have used the abbreviations

$$\tilde{u} = \frac{1}{2u}, \quad \tilde{v} = \frac{1}{2v}$$

We can combine these maps to 16 different transformations $(u, v) - (u', v')$, or in other words, we found 16 different coordinate systems which are all associated to the same choice of gauge (31). Applying the transformations to the field equations with modulus signs removed ((34)-(36)) we find, that the 16 transformations directly correspond to the 16 cases described by s_1-s_4. In particular, if we (arbitrarily) choose the solution with $(s_1, s_2, s_3, s_4) = (1, 1, 1, 1)$ in the (u, v)-coordinate system, then all the other 15 solutions are reproduced by applying all the 15 nontrivial coordinate transformations to the initially chosen solution (see Table 3).

3.6 Local integration, part 2

Now we continue the integration process in the case $C \neq 0$. First we define a new binary switch, which represents the sign of C:

$$s_c = \frac{C}{|C|}$$

$(ss_2, s_c) = (1,1)$	$(ss_2, s_c) = (-1,1)$
$F_1(\xi)$: $]-\infty,\infty[\to]-\infty,-\frac{C}{2}[$	$F_1(\xi)$: $]-\infty,\infty[\to]-\infty,-\frac{C}{2}[$
$F_2(\xi)$: $]\xi_2(0),\infty[\to]-\frac{C}{2},0[$	$F_2(\xi)$: $]-\infty,\xi_2(0)[\to]-\frac{C}{2},0[$
$F_3(\xi)$: $]\xi_3(0),\infty[\to]0,\infty[$	$F_3(\xi)$: $]-\infty,\xi_3(0)[\to]0,\infty[$

$(ss_2, s_c) = (1,-1)$	$(ss_2, s_c) = (-1,-1)$
$F_1(\xi)$: $]-\infty,\infty[\to]-\frac{C}{2},\infty[$	$F_1(\xi)$: $]-\infty,\infty[\to]-\frac{C}{2},\infty[$
$F_2(\xi)$: $]-\infty,\xi_2(0)[\to]0,-\frac{C}{2}[$	$F_2(\xi)$: $]\xi_2(0),\infty[\to]0,-\frac{C}{2}[$
$F_3(\xi)$: $]-\infty,\xi_3(0)[\to]-\infty,0[$	$F_3(\xi)$: $]\xi_3(0),\infty[\to]-\infty,0[$

Table 4: Domain and image of $F_1(\xi)$-$F_3(\xi)$ in various cases

The field equation (46) can be integrated:

$$\frac{dF}{d\xi} = ss_2\left(1+\frac{C}{2F}\right) \quad \Rightarrow \quad \frac{d\xi}{dF} = ss_2\frac{2F}{2F+C} \tag{56}$$

$$\Rightarrow \quad \xi(F) = ss_2\left(F - \frac{C}{2}\log|2F+C| + C'\right) \tag{57}$$

The integration constant C' again depends on the binary switches s_1-s_4, but in this case it can also depend on C. It is not possible to solve equation (57) explicitly for F, but we can still get valuable information from (56) and (57). Equation (56) shows, that there are two critical points: $F = 0$ and $F = -C/2$. These critical points split the domain of $\xi(F)$ into three intervals, so we can define three functions $\xi_1(F), \xi_2(F), \xi_3(F)$ by restricting $\xi(F)$ to each of the intervals. Each of these functions is a diffeomorphism, we denote their inverses with $F_1(\xi), F_2(\xi), F_3(\xi)$. Since the images of $\xi_1(F), \xi_2(F), \xi_3(F)$ can be obtained easily, we can describe the domains and the images of $F_1(\xi), F_2(\xi), F_3(\xi)$ as well. Table 4 shows the results of this procedure.

In each case there exist three different inequivalent solutions of the field equation (46). We define binary switches in order to characterize these solutions:

$$s_5 = s_c\frac{F}{|F|}, \quad s_6 = -s_c\frac{2F+C}{|2F+C|} \tag{58}$$

The combination $(s_5, s_6) = (1,1)$ is forbidden. This follows from

$$s_6 = -\frac{|2F|s_c\frac{F}{|F|}+|C|}{|2F+C|} = -\frac{s_5|2F|+|C|}{|2F+C|}$$

(s_5, s_6)		$F_i(\xi)$
$(-1,\ \ 1)$	$\leftrightarrow$	$F_1(\xi)$
$(-1, -1)$	$\leftrightarrow$	$F_2(\xi)$
$(\ \ 1, -1)$	$\leftrightarrow$	$F_3(\xi)$

Table 5: The relation between (s_5, s_6) and $F_1(\xi)$-$F_3(\xi)$

A quick investitation reveals the relationship between the switches s_5,s_6 and the functions $F_i(\xi)$, as shown in Table 5. With the help of s_6 we can get rid of the modulus sign in expression (57) for $\xi(F)$. Let us summarize the results we have derived so far:

$$\begin{aligned} \xi(F) &= ss_2\left(F - \frac{C}{2}\log(-s_6 s_c(2F+C)) + C'\right) && (59) \\ \omega^2 &= ss_1s_2\frac{1}{2\eta^2 uv}\left(1+\frac{C}{2F}\right) && (60) \\ ds^2 &= s\omega^2(dudv + dvdu) + F^2 d\Omega^2 && (61) \\ &= s_1s_2\left(1+\frac{C}{2F}\right)(-dt^{*2} + dr^{*2}) + F^2 d\Omega^2 && (62) \end{aligned}$$

Choosing appropriate coordinate systems

In section 3.3 we introduced the (r^*, t^*)-coordinates in order to integrate the field equations, and this finally lead to the expressions (59) and (60), which are presented in a mixture of different coordinates. In order to express the local solutions in the (u, v)-coordinates we need to transform (59).

Now we have an additional freedom concerning the (u, v)-coordinates. As we have seen in section 3.5 when talking about the symmetries of the field equations, there are coordinate transformations which transform the different kinds of solutions represented by s_1, s_2, s_3 and s_4 into eachother and which preserve the gauge choice. So the question arises how to assign these coordinate systems to the various cases. Our answer is to assign the coordinates in such a way that every solution should "look the same", which will have the effect that the formulas in the (u, v)-coordinates will get rid of the binary switches s, s_1 and s_2. We define new coordinates (u', v') as follows:

$$(u', v') = \begin{cases} (u, v) & \text{if } s_1s_2 = \ \ 1,\ ss_2s_c = -1 \\ (\tilde{u}, \tilde{v}) & \text{if } s_1s_2 = \ \ 1,\ ss_2s_c = \ \ 1 \\ (u, \tilde{v}) & \text{if } s_1s_2 = -1,\ ss_2s_c = -1 \\ (\tilde{u}, v) & \text{if } s_1s_2 = -1,\ ss_2s_c = \ \ 1 \end{cases} \qquad (63)$$

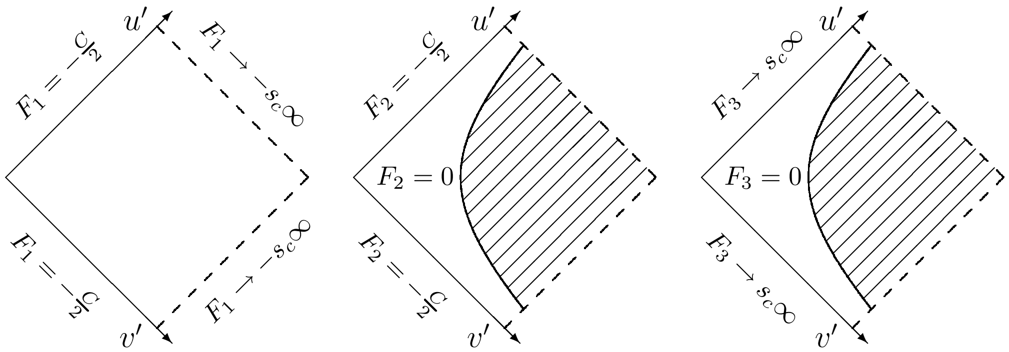

Figure 1: Domain and image of $F_1(u',v')$-$F_3(u',v')$

Here we have again used the abbreviations

$$\tilde{u} = 1/2u, \quad \tilde{v} = 1/2v \tag{64}$$

The transformations (63) together with the formulas (39) and (57) allow us to determine the domain and image of the three functions $F_1(u',v')$-$F_3(u',v')$, as shown in Figure 1. Quantitatively, we can use these formulas to find a relation between ξ and $2u'v'$:

$$\left.\begin{array}{llll} s_1s_2 = & 1, & ss_2s_c = -1: & 2u'v' = s_3s_4e^{2\eta r^*} \\ s_1s_2 = & 1, & ss_2s_c = 1: & 2u'v' = s_3s_4e^{-2\eta r^*} \\ s_1s_2 = & -1, & ss_2s_c = -1: & 2u'v' = s_3s_4e^{2\eta t^*} \\ s_1s_2 = & -1, & ss_2s_c = 1: & 2u'v' = s_3s_4e^{-2\eta t^*} \end{array}\right\} \Rightarrow \; 2u'v' = s_3s_4e^{-ss_2s_c2\eta\xi}$$

Inserting the expression (59) for $\xi(F)$ we get $F(u',v')$ in implicit form:

$$2u'v' = -s_3s_4s_6s_ce^{-s_c2\eta(F+C')}(2F+C)^{\eta|C|} \tag{65}$$

Now we transform the metric (61) to the new coordinates:

$$s'\omega'^2(du'dv' + dv'du') + F^2d\Omega^2 = s\omega^2(dudv + dvdu) + F^2d\Omega^2$$

Using (60),(63) and (64) we get

$$\frac{d\tilde{u}}{du} = -\frac{\tilde{u}}{u}, \quad \frac{d\tilde{v}}{dv} = -\frac{\tilde{v}}{v} \quad \Rightarrow \quad \frac{du'}{du}\frac{dv'}{dv} = s_1s_2\frac{u'v'}{uv}$$

$$\Rightarrow \quad s'\omega'^2 = s\omega^2\frac{du}{du'}\frac{dv}{dv'} = s\omega^2s_1s_2\frac{uv}{u'v'} = \frac{1}{2\eta^2u'v'}(1+\frac{C}{2F})$$

and the insertion of the expression (65) for $2u'v'$ leads to a formula for $\omega'^2(F)$, which does not depend explicitly on u' and v' anymore:

$$\begin{aligned} s'\omega'^2 &= -s_3 s_4 s_6 s_c \frac{1}{\eta^2} e^{s_c 2\eta(F+C')} (2F+C)^{-\eta|C|} (1 + \frac{C}{2F}) \\ &= -s_3 s_4 s_6 s_c \frac{1}{2F\eta^2} e^{s_c 2\eta(F+C')} (2F+C)^{-\eta|C|+1} \end{aligned}$$

Now the results can be summarized. We write down the solutions in the (u', v')- and in the (r^*, t^*)-coordinates:

$$2u'v'(F) = -s_3 s_4 s_6 s_c e^{-s_c 2\eta(F+C')} (2F+C)^{\eta|C|} \tag{66}$$

$$ds^2 = -s_3 s_4 s_6 s_c \frac{1}{2F\eta^2} e^{s_c 2\eta(F+C')} (2F+C)^{-\eta|C|+1} \times (du'dv' + dv'du') + F^2 d\Omega^2 \tag{67}$$

$$\xi(F) = s s_2 \left(F - \frac{C}{2} \log(-s_6 s_c (2F+C)) + C' \right) \tag{68}$$

$$ds^2 = s_1 s_2 \left(1 + \frac{C}{2F} \right) (-dt^{*2} + dr^{*2}) + F^2 d\Omega^2 \tag{69}$$

It is worth noting, that no assumptions other than spherical symmetry were made, therefore the formulas (66) and (67) deliver in each domain of integration I-IV all local spherically symmetric solutions of the vacuum field equations. The formulas (68) and (69) do the same, but the domain of integration in these coordinates corresponds to only one domain of integration I-IV in the (u', v')-coordinates.

4 Construction of the global solutions

In this section we explicitly construct all maximal global solutions. In the (u', v')-coordinates we have four domains of integration, each with four edges, where the solution eventually can be continued. These edges fall into two categories:

- edges with $u' = 0$ or $v' = 0$
- edges with $u' = \infty$ or $v' = \infty$

The situation is clear for the edges of the former category: for a given edge we have to find all local solutions which are regular on this edge and then check for all neighbouring pairs of solutions, whether they can be glued together, satisfying the field equations on the edge.

For the edges of the latter category, the situation is different, because the (u', v')-coordinates do not cover these edges. Fortunately, as we will see, we do not have to consider these edges at all, because we will already obtain all maximal solutions by only considering the edges of the former category. This is a consequence of the way we defined the (u', v')-coordinates in (63).

4.1 The gluing process

Now we glue local solutions together, only considering the edges with $u' = 0$ resp. $v' = 0$. From equation (66) we see, that on the edge we have either $F = s_c\infty$ or $F = -C/2$. The definition of s_5 (58) then implies:

$$\begin{array}{rcll} s_5 = 1 & \Rightarrow & F = s_c\infty & \text{on the edge} \\ s_5 = -1 & \Rightarrow & F = -\frac{C}{2} & \text{on the edge} \end{array}$$

In the case $s_5 = 1$ equation (67) implies, that the metric in (u', v')-coordinates is not regular on the edge. Therefore we only have to consider the solutions with $s_5 = -1$ for the gluing process. In the following, we use the term *global solution* for all solutions obtained by the gluing process described above, even if we don't know yet, whether these solutions are maximal, since we do not consider the edges with $u' = \infty$ and $v' = \infty$. In this sense, the local solutions with $s_5 = 1$ are considered as global solutions.

Well-definedness of the (u', v')-coordinates

The first step is to find out, whether local solutions with different (u', v')-coordinates can be glued together. The answer is certainly negative in the case that both coordinates differ, since such local solutions do not share a common edge with $u' = 0$ or $v' = 0$. Now let there be two neighbouring local solutions where one coordinate is different and the other one is not. Without loss of generality we assume, that the v'-coordinate is different. In these coordinates, F and the metric are constant on the edge for both solutions, as can be seen in the equations (66) and (67). To discuss the continuity of the metric on the edge we need to transform one solution, so that both solutions have the same coordinates. Thus one metric acquires a factor $\partial v'_2/\partial v'_1 = -v'_2/v'_1 = -1/2{v'_1}^2$, making it non-constant on the edge. Since the other metric is constant on the edge, there is no chance for continuity. We conclude that only local solutions having the same (u', v')-coordinates can be glued together. In the subsequent discussion we always assume neighbouring local solutions to have the same coordinates.

Well-definedness of η and $|C|$

Let there be two neighbouring local solutions. Looking at equation (32) we see, that both solutions must have the same value for η if the metric is to be $\mathcal{C}^2$ on the common edge. so η is well-defined on every global solution.

Since we are only considering the solutions with $s_5 = -1$, we have $F = -C/2$ on the common edge. Therefore we see from equation (67), that any of the metrics vanishes on the edge, if $\eta \neq 1/|C|$. Thus we need to fix the gauge freedom η with

$$\eta = \frac{1}{|C|}$$

Since η is well-defined for every global solution, the same also applies to $|C|$. The local solutions now read

$$ds^2 = -s_3 s_4 s_6 s_c \frac{C^2}{2F} e^{\frac{2(F+C')}{C}} (du'dv' + dv'du') + F^2 d\Omega^2 \qquad (70)$$

$$2u'v'(F) = -s_3 s_4 s_6 s_c e^{-\frac{2(F+C')}{C}} (2F + C) \qquad (71)$$

Well-definedness of C, C' and $F/|F|$

Now we focus on the integration constant C'. Again we consider a common edge of two local solutions. We evaluate the metric on the edge:

$$ds^2(F = -\frac{C}{2}) = s_3 s_4 s_6 |C| e^{\frac{2C'}{C} - 1} (du'dv' + dv'du') + F^2 d\Omega^2 \qquad (72)$$

As we can see, a necessary condition for two local solutions to be glued together is

$$\frac{C'_1}{C'_2} = \frac{C_1}{C_2} \qquad (73)$$

so $|C|$ and $|C'|$ are well-defined for each global solution.

Up to this point, C and C' of two neighbouring local solutions can have different signs. We note however, that the equations (70) and (71) are invariant under simultaneous sign reversal of C, C' and F. Therefore given a local solution we can always find an identical local solution with C sign-reversed. So we can assume C to be well-defined on every global solution. Then the same applies to C' due to equation (73) and $s_5 F/|F|$ is also well-defined due to (58). But since the solutions with $s_5 = 1$ do not participate in the gluing process, s_5 is also well-defined. Therefore we can assume C, C' and $F/|F|$ to be well-defined. We can even choose the sign of one entity arbitrarly for every global solution. Our choice is:

$$F > 0 \qquad (74)$$

This choice also fixes the sign of C due to (58). The equations (70) and (71) then imply, that for every choice of C and C' there exist at most three inequivalent local solutions in every domain of integration, namely those described by all the allowed combinations of (s_5, s_6). Since we do not glue the solutions with $s_5 = 1$, there are only two solutions left to be considered for the gluing process.

Gluing the local solutions together

Having a second look at the metric evaluated on the edge (72) we find another condition, which neighbouring local solutions have to satisfy in order to be glued together: $(s_3 s_4 s_6)_1 = (s_3 s_4 s_6)_2$. Since neighbouring solutions have $(s_3 s_4)_1 \neq (s_3 s_4)_2$, the condition implies $(s_6)_1 \neq (s_6)_2$. It follows, that in the case $s_5 = -1$ there exist at most two global solutions with s_6 alternating when going to a neighbouring domain. The local solutions to be glued together read

$$
\begin{aligned}
ds^2 &= \tilde{s}\frac{C^2}{2F}e^{\frac{2(F+C')}{C}}(du'dv' + dv'du') + F^2 d\Omega^2 && (75)\\
2u'v'(F) &= \tilde{s}e^{-\frac{2(F+C')}{C}}(2F + C), \quad C < 0, \quad F > 0 && (76)
\end{aligned}
$$

where the (globally well-defined) binary switch $\tilde{s}$ reflects the two cases mentioned above. We see that the local solutions to be glued together are represented by the same two analytic expressions, both being regular on the edge. So we can define the metric on all points with $u' = 0$ and $v' = 0$ by evaluating (75) at these points, yielding a metric defined on the whole (u', v')-plane which is $\mathcal{C}^2$ on all added points. Then the Ricci tensor is continous everywhere, so that the field equations are certainly satisfied on all added points.

In the case $s_5 = 1$ we have $s_6 = -1$ and $C > 0$ due to (74). Again we get two global solutions, which, depending on the domain of integration, differ by a sign. The global solutions in the case $s_5 = 1$ take the form:

$$
\begin{aligned}
ds^2 &= \tilde{s}\frac{C^2}{2F}e^{\frac{2(F+C')}{C}}(du'dv' + dv'du') + F^2 d\Omega^2 && (77)\\
2u'v'(F) &= \tilde{s}e^{-\frac{2(F+C')}{C}}(2F + C), \quad C > 0, \quad F > 0 && (78)\\
\tilde{s} &= \begin{cases} -1 & \text{(I,III)} \\ \;\;\,1 & \text{(II,IV)} \end{cases}
\end{aligned}
$$

The integration constant C' represents a gauge freedom. This can be seen by considering the transformation $(u'', v'') = (\exp(C'/C)u', \exp(C'/C)v')$ and using the same argument as the one given in section 3.3 on page 8 for the gauge freedom concerning the functions $C_1(v)$ and $C_2(u)$. We fix the gauge freedom with

$$C' = \frac{C}{2}\log|C| \tag{79}$$

The two global solutions described by $\tilde{s}$ are equivalent in both cases $s_5 = -1$ and $s_5 = 1$, which can be seen by transforming the metric by $(u', v') \to (-u', v')$. Therefore we select a sign and dispose of the binary switch $\tilde{s}$. By joining the two cases $s_5 = -1$ and $s_5 = 1$, choosing the original (u, v)-coordinates for the metric and inserting the gauge (79) for C' we get:

$$\begin{aligned} ds^2 &= -\frac{C^3}{2F} e^{\frac{2F}{C}} (dudv + dvdu) + F^2 d\Omega^2 & (80) \\ 2uv(F) &= -e^{-\frac{2F}{C}} \left(1 + \frac{2F}{C}\right), \quad F > 0 & (81) \\ uv &\in \begin{cases}]-\infty, \infty[& \text{if } C < 0 \\ ]0, \infty[& \text{if } C > 0 \end{cases} & (82) \end{aligned}$$

These formulas describe all global spherically symmetric solutions of the vacuum field equations with $C \neq 0$, which can be covered by the (u, v)-coordinate system. The third line states, that the formulas (80) and (81) in the case $C > 0$ only satisfy the field equations in the domains I and III. The results shown in Figure 1 combined with the condition $F > 0$ allow us to write down the image of $F(u, v)$ as follows:

$$F(u,v) \in \begin{cases}]-C/2, \infty[& \text{if } (u,v) \in \{\text{I,III}\}, \quad C < 0 \\ ]0, -C/2[& \text{if } (u,v) \in \{\text{II,IV}\}, \quad C < 0 \\ ]0, \infty[& \text{if } (u,v) \in \{\text{I,III}\}, \quad C > 0 \end{cases} \tag{83}$$

4.2 Maximality of the global solutions

In section 3.4 we gave a definition of maximal spacetimes. Less rigorously formulated a spacetime is maximal, when no geodesic reaches the border of the spacetime at a finite value of the affine parameter, with the exception of the cases where the geodesic runs into a true spacetime singularity. In the following we prove the maximality of all global solutions derived so far.

The geodesics on the event horizon

First we analyse the null geodesics in the case $C \neq 0$. The geodesic equations are generated by the Lagrangian

$$L = -\frac{C^3}{F} e^{\frac{2F}{C}} \frac{du}{d\lambda}\frac{dv}{d\lambda} + F^2 \left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2(\theta) \left(\frac{d\phi}{d\lambda^2}\right)^2 \right)$$

which follows from (80). With λ we denote the affine parameter of the geodesics. Due to spherical symmetry we do not have to analyse geodesics with $d\theta/d\lambda \neq 0$ or $d\phi/d\lambda \neq 0$ because they do not contribute to the question of maximality. By setting $d\theta/d\lambda = d\phi/d\lambda = 0$ the geodesic equations for $u(\lambda)$ and $v(\lambda)$ read:

$$\frac{d}{d\lambda}\left(-\frac{C^3}{F}e^{\frac{2F}{C}}\frac{du}{d\lambda}\right) = \frac{\partial}{\partial v}\left(-\frac{C^3}{F}e^{\frac{2F}{C}}\frac{du}{d\lambda}\frac{dv}{d\lambda}\right) \tag{84}$$

$$\frac{d}{d\lambda}\left(-\frac{C^3}{F}e^{\frac{2F}{C}}\frac{dv}{d\lambda}\right) = \frac{\partial}{\partial u}\left(-\frac{C^3}{F}e^{\frac{2F}{C}}\frac{du}{d\lambda}\frac{dv}{d\lambda}\right) \tag{85}$$

The Lagrangian itself is a constant of motion along geodesics:

$$L(\lambda) = -\frac{C^3}{F}e^{\frac{2F}{C}}\frac{du}{d\lambda}\frac{dv}{d\lambda} = \text{const}$$

This leads to the following relation:

$$L(\lambda) = 0 \quad \Leftrightarrow \quad du/d\lambda = 0 \quad \text{or} \quad dv/d\lambda = 0 \tag{86}$$

So we find that all null geodesics have $u(\lambda) = \text{const}$ or $v(\lambda) = \text{const}$. Using the equations (84) and (85) we get

$$\frac{du}{d\lambda} = 0 \quad \Rightarrow \quad \frac{d}{d\lambda}\left(-\frac{C^3}{F}e^{\frac{2F}{C}}\frac{dv}{d\lambda}\right) = 0 \quad \Rightarrow \quad \frac{dv}{d\lambda} = -\frac{\alpha F}{C^3}e^{-\frac{2F}{C}} \tag{87}$$

$$\frac{dv}{d\lambda} = 0 \quad \Rightarrow \quad \frac{d}{d\lambda}\left(-\frac{C^3}{F}e^{\frac{2F}{C}}\frac{du}{d\lambda}\right) = 0 \quad \Rightarrow \quad \frac{du}{d\lambda} = -\frac{\beta F}{C^3}e^{-\frac{2F}{C}} \tag{88}$$

with the integration constants α and β. Now let $p \in H$ where H is the set of all points with $u = 0$ or $v = 0$, called *event horizon*. Since $F = -C/2$ on H the equations (87) and (88) reduce to

$$\frac{du}{d\lambda}(p) = 0 \quad \Rightarrow \quad \frac{dv}{d\lambda}(p) = \alpha'(C) = \text{const on } H \tag{89}$$

$$\frac{dv}{d\lambda}(p) = 0 \quad \Rightarrow \quad \frac{du}{d\lambda}(p) = \beta'(C) = \text{const on } H \tag{90}$$

Now we can summarize the first result of our geodesic analysis. Every geodesic, which at p goes tangent to H is a null geodesic according to (86). It is described by the expressions $(u, v)(\lambda) = (0, \alpha'(C)\lambda)$ resp. $(u, v)(\lambda) = (\beta'(C)\lambda, 0)$, which state that the geodesic runs forever on H with respect to the affine parameter. On the other hand, all other geodesics, which at p do not go tangent to H, leave H and enter one of the domains I-IV.

The geodesics initiating outside the event horizon

The next step is to consider the geodesics starting somewhere outside the event horizon, in one of the domains I-IV. The subsequent geodesic analysis is mainly based on [5].

We start with the local solutions in (r^*, t^*)-coordinates, given by the equations (46) and (49). These formulas contain the binary switches s, s_1 and s_2, which we want to fix first. We note, that the expressions $s_1 s_2$ and $s s_2$ are well-defined for every global solution. To see this, consider the coordinate transformations defined in (63). As discussed in section 4.1 on page 19, only local solutions having the same primed coordinates can be part of the same global solution. Since s_c is well-defined for every global solution, our assertion then follows from (63). We choose $s_1 s_2 = 1$ and $s s_2 = 1$.

The equation (46) defines a coordinate transformation $\xi \leftrightarrow F$ in every domain I-IV, as explained in section 3.6 on page 15. So we can transform the metric (49) to the new coordinates:

$$dr^* = \left(1 + \frac{C}{2F}\right)^{-1} dF$$

$$\Rightarrow \quad ds^2 = -\left(1 + \frac{C}{2F}\right) dt^{*2} + \left(1 + \frac{C}{2F}\right)^{-1} dF^2 + F^2 d\Omega^2 \tag{91}$$

Now we start analysing the geodesics. Again we do not have to consider the angular motions, thus we set $d\theta/d\lambda = d\phi/d\lambda = 0$, where λ is the affine parameter of the geodesics. The geodesic equations are generated by the Lagrangian

$$L = -\left(1 + \frac{C}{2F}\right)\left(\frac{dt^*}{d\lambda}\right)^2 + \left(1 + \frac{C}{2F}\right)^{-1}\left(\frac{dF}{d\lambda}\right)^2$$

Again L is a constant of motion, which gives us an additional differential equation:

$$-\left(1 + \frac{C}{2F}\right)\left(\frac{dt^*}{d\lambda}\right)^2 + \left(1 + \frac{C}{2F}\right)^{-1}\left(\frac{dF}{d\lambda}\right)^2 = \alpha \tag{92}$$

According to our signature convention we have

$$\begin{array}{ll} \alpha < 0 & \text{if } (t^*, F)(\lambda) \text{ is timelike} \\ \alpha = 0 & \text{if } (t^*, F)(\lambda) \text{ is null} \\ \alpha > 0 & \text{if } (t^*, F)(\lambda) \text{ is spacelike} \end{array}$$

Since L does not depend explictly on t we have an additional constant of motion:

$$\left(1 + \frac{C}{2F}\right)\frac{dt^*}{d\lambda} = E \tag{93}$$

Inserting (93) into (92) we get the following ordinary differential equation for $dF/d\lambda$:

$$\left(\frac{dF}{d\lambda}\right)^2 - \alpha\left(1 + \frac{C}{2F}\right) = E^2 \tag{94}$$

Formal integration leads to the following integral expression for $\lambda(F)$:

$$\lambda = \int_{F_0}^{F} \frac{dF'}{\sqrt{E^2 + \alpha\,(1 + C/2F')}}, \quad F_0 = F(\lambda = 0) \tag{95}$$

We do not have to carry out the integral, since we are only interested in the question, whether any geodesic reaches the border of the spacetime at a finite value of the affine parameter. In each domain the boundary of the local solution is either $F = \infty$, $F = -C/2$ or $F = 0$, as can be seen from Figure 1, remembering the condition $F > 0$.

The singularity at $F = 0$ is a true spacetime singularity. This can be seen by calculating the curvature invariant $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 3C^2/F^6$. This invariant clearly diverges for $F \to 0$, while it approaches a finite value for $F \to -C/2$. Concerning the question of maximality we can forget about the case $F = 0$, since the geodesics are allowed to run into true spacetime singularities.

The case $F = -C/2$ can also be handled quickly: as we have seen previously, each geodesic, which passes through a point on the event horizon, either runs forever on the horizon or enters another domain and continues to run there. So to prove the maximality of the spacetime, we only need to show, that there does not exist any geodesic that reaches $F = \infty$ at a finite value of the affine parameter.

Let p be any point on the spacetime with $F = F_0 \neq -C/2$. Then equation (95) describes all geodesics through p. We only consider those geodesics which are not bounded from above with respect to the F-coordinate. The existence of such geodesics implies the condition $E^2 + \alpha \geq 0$, since otherwise the argument of the square root would become negative in the limit $F \to \infty$. In the case $E^2 + \alpha = 0$ the integral evaluates to

$$\lambda = \int_{F_0}^{F} \frac{dF'}{\sqrt{(\alpha C/2F')}} = \frac{2}{3}\sqrt{\frac{2}{\alpha C}}\left(F^{\frac{3}{2}} - F_0^{\frac{3}{2}}\right)$$

and it follows $\lambda(F \to \infty) \to \infty$, so the border at $F = \infty$ is never reached at a finite value of λ. Now we consider the case $E^2 + \alpha > 0$. When replacing the term $C/2F'$ in (95) with 0 resp. $C/(2F_0)$ both integrals diverge in the limit $F \to \infty$ and it follows, that the same happens to the original integral. Thus we have shown that no geodesic in the whole spacetime ever reaches $F = \infty$ at a finite value of the affine parameter. This finalizes the proof concerning the maximality of all global solutions derived so far. Note, that this proof basically also applies to the case $C = 0$ with only minor adaptations.

4.3 Associating the mass

The global solutions described by the equations (80)-(82) still contain the integration constant C, which naturally should be associated to the mass of the system. There exist several mass definitions, which all require the spacetime to be *asymptotically flat*, which roughly means, that the geometry of the spacetime becomes Minkowskian at spatial infinity. For a rigorous definition of asymptotical flatness, see [4] for example.

In each domain the metric of our global solutions can be transformed to (t^*, F)-coordinates, which yields the equation (91) derived in section 4.2. The form of the metric and the positive sign of F leads to the interpretation of F as *radial coordinate*, while t^* represents a time coordinate. Thus we rename the coordinates to reflect these interpretations:

$$g = ds^2 = -\left(1 + \frac{C}{2r}\right) dt^2 + \left(1 + \frac{C}{2r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (96)$$

In the limit $r \to \infty$ (96) converges towards the Minkowski metric, showing that all global solutions are indeed asymptotically flat. Additionally they are *stationary* in all local domains with $s_5 s_6 = -1$, which means that there exist coordinates, where the metric does not depend on the timelike coordinate. The Minkowski spacetime and the global solutions with $C > 0$ are globally stationary, while the global solutions with $C < 0$ are only stationary for $r > -C/2$, outside the event horizon.

In the following we set both the gravitational constant and the velocity of light to one: $G = c = 1$. For stationary, asymptotically flat spacetimes the *Komar mass* is defined as [6]:

$$m = -\frac{1}{8\pi} \int_{S^2_\infty} *dk \qquad (97)$$

Here k denotes the one-form associated to the Killing field ∂_t, $*$ denotes the Hodge operator and the integration is carried out over a sphere at spatial infinity. We want to calculate the relationship between m and the (globally well-defined) integration constant C. To use the formula (97) in case of $C < 0$, we have to restrict ourselves to the exterior region of the black hole. Both this exterior region and the global spacetime with $C > 0$ are described by the metric (96) with $1 + C/2r > 0$. Now we need two intermediate results:

$$k(\partial_t) = g(\partial_t, \partial_t) = -\left(1 + \frac{C}{2r}\right) \;\Rightarrow\; k = -\left(1 + \frac{C}{2r}\right) dt$$

$$*(dr \wedge dt) = \sqrt{|\det g|}(d\theta \wedge d\phi) = r^2 \sin\theta (d\theta \wedge d\phi) = r^2 d\Omega$$

The calculation now goes as follows:

$$\begin{aligned} m &= -\frac{1}{8\pi}\int_{S^2_\infty} *d\left(-\left(1+\frac{C}{2r}\right)dt\right) = -\frac{1}{8\pi}\int_{S^2_\infty} *\left(\frac{C}{2r^2}dr \wedge dt\right) \\ &= -\frac{1}{8\pi}\int_{S^2_\infty} \frac{C}{2}d\Omega = -\frac{C}{4} \end{aligned}$$

Thus we have found the following relationship between m and C:

$$C = -4m$$

The metric (96) now turns into the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{98}$$

which reduces to the Minkoswki metric when the mass is set to zero. Rewriting the global solutions (80)-(82) leads to the Kruskal metric describing the Kruskal-Schwarzschild manifold:

$$ds^2 = \frac{32m^3}{r} e^{-\frac{r}{2m}}(dudv + dvdu) + r^2 d\Omega^2 \tag{99}$$

$$2uv(r) = e^{\frac{r}{2m}}\left(\frac{r}{2m} - 1\right), \quad r > 0 \tag{100}$$

$$uv \in \begin{cases}]-\infty, \infty[& \text{if } m > 0 \\ ]0, \infty[& \text{if } m < 0 \end{cases} \tag{101}$$

The solutions with positive mass describe the Schwarzschild black hole with a singularity shielded by an event horizon, while the solutions with negative mass represent a spacetime with a naked singularity (see Figure 2). These mass-parametrized solutions and the Minkowski spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad m = 0 \tag{102}$$

are all maximal spherically symmetric solutions of the vacuum Einstein equations with zero cosmological constant.

In the gauge (31) we have proven that these solutions are the only $\mathcal{C}^2$-solutions: we did not use analytic continuation, nor did we make any arbitrary choices other than the gauge choice. To rigorously prove this result in a gauge-independent manner seems difficult, since the gauge can been chosen independently in every domain of integration.

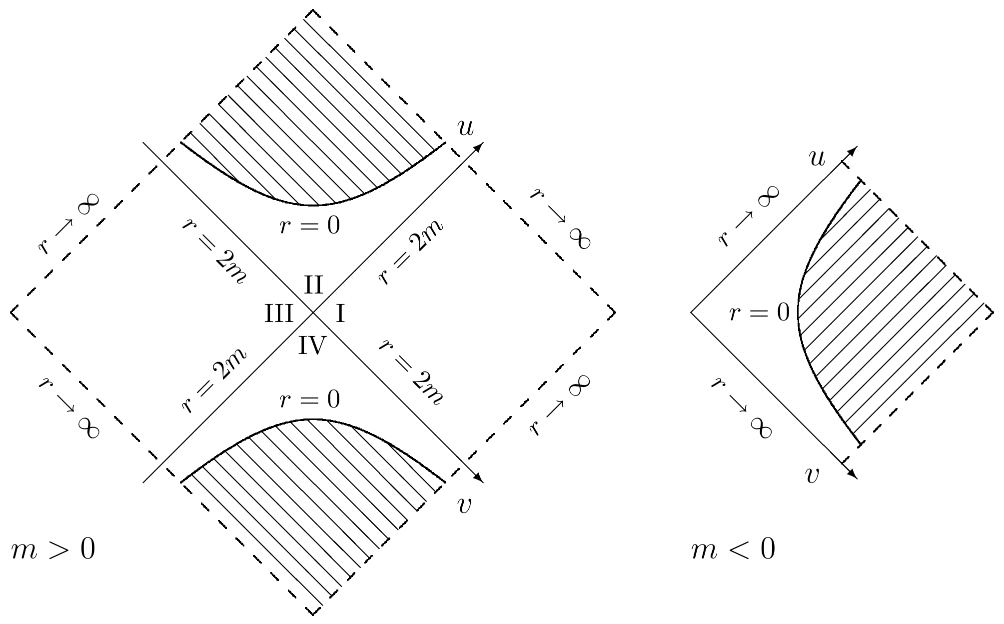

Figure 2: Diagrams of the maximal solutions for $m \neq 0$

5 Summary

In this paper we derived all maximal spherically symmetric $\mathcal{C}^2$-solutions of the vacuum Einstein equations with zero cosmological constant. These solutions turn out to be the Minkowski spacetime and the mass-parametrized family of Kruskal-Schwarzschild spacetimes, where the mass a priori can have both signs. The spacetimes with negative mass are usually considered unphysical, because moving observers experience repulsive gravitational forces, and this was never observed up to now. Additionally these spacetimes contain naked singularities which are believed not to exist (this is known as *cosmic censorship conjecture*).

There are two different goals we wanted to achieve by writing this paper. First we wanted to show, that one doesn't need the theorem of analytic continuation in order to derive all maximal solutions. Our derivation does not favor any local solution, in contrast to the historical approaches, which first derived the Schwarzschild solution, followed by analytic continuation. This should remind us, that there is no reason to treat the opposing local solutions (such as the black hole/white hole pair or the two parallel universes in the positive mass Kruskal-Schwarzschild manifold) differently from a global perspective.

Our second goal was to offer a thorough case study for students learning general relativity. The derivation shown in this paper covers all the steps of the whole integration process, avoiding the use of complicated techniques and nontrivial theorems as much as possible.

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A Appendix

Here we give some proofs of propositions used in our work.

A.1 Pseudo-Riemannian Surfaces[3]

Proposition 1 *The Einstein tensor vanishes on every pseudo-riemannian surface*

Proof. Let g_{ij} $(i, j \in \{0, 1\})$ be the components of the metric tensor, let G_{ij} be the Einstein tensor. Choose any point p on the surface and normal coordinates with origin at p. In these coordinates we have

$$g_{ij}(p) = \mathrm{diag}(\pm 1, \pm 1), \quad \Gamma^k{}_{ij}(p) = 0 \quad \text{and} \quad g_{ij,k}(p) = 0$$

The calculation of the Ricci tensor components and the Ricci scalar yields

$$\begin{aligned}
R_{00}(p) &= \frac{1}{2} g^{11}(p)(-g_{00,1,1}(p) + 2g_{01,0,1}(p) - g_{11,0,0}(p)) \\
R_{11}(p) &= \frac{1}{2} g^{00}(p)(-g_{00,1,1}(p) + 2g_{01,0,1}(p) - g_{11,0,0}(p)) \\
R_{01}(p) &= R_{10}(p) = 0 \\
R(p) &= g^{00}(p) g^{11}(p)(-g_{00,1,1}(p) + 2g_{01,0,1}(p) - g_{11,0,0}(p))
\end{aligned}$$

Using the relations $1 = \delta^0_0 = g_{00}(p) g^{00}(p)$ and $1 = \delta^1_1 = g_{11}(p) g^{11}(p)$ we get

$$R_{ij}(p) = \frac{1}{2} g_{ij}(p) R(p) \quad \Rightarrow \quad G_{ij}(p) = R_{ij}(p) - \frac{1}{2} g_{ij}(p) R(p) = 0$$

and the arbitrary choice of p finalizes the proof. □

To prove the next proposition we need a result concerning integrating factors of one-forms:

[3]With the term *surface* we always mean twodimensional manifolds

Lemma 1 *Let θ be a $\mathcal{C}^2$-one-form on a pseudo-riemannian surface, nowhere vanishing, then there locally exist $\mathcal{C}^2$-functions λ and u so that $\theta = \lambda du$.*

Proof. Let p be any point on the surface, let (x, y) be an arbitrary local coordinate system centered at p. Then θ has the form

$$\theta = A(x, y)dx + B(x, y)dy$$

where A and B are $\mathcal{C}^2$-functions. If either A or B vanishes at p, then we can define new coordinates using $(x, y) = (x' + y', x' - y')$, leading to $\theta = A'(x', y')dx' + B'(x', y')dy'$ where A' and B' do not vanish at p. Therefore we can safely assume A and B to be both nonvanishing at p resp. in a neighbourhood of p.

With $\theta = 0$ we denote an associated ordinary differential equation (ODE)

$$\frac{dy}{dx} = -\frac{A(x, y)}{B(x, y)}$$

which, by the existence and uniqueness theorem, locally has a unique solution of the form $u(x, y(x)) = c$, which implies

$$du = \partial_x u dx + \partial_y u dy = 0$$

where $\partial_x u$ and $\partial_y u$ are both $\mathcal{C}^2$-functions. Since the ODE's generated by θ and du have the same solutions, the following condition must hold:

$$\frac{A(x, y)}{B(x, y)} = \frac{\partial_x u(x, y)}{\partial_y u(x, y)}$$

which implies that there locally exists a $\mathcal{C}^2$-function $\lambda(x, y)$ with $\theta = \lambda du$. $\square$

Proposition 2 *Every pseudo-riemannian $\mathcal{C}^2$-surface with signature (1,1) is conformally flat, which means that for every point p on the surface there exist local coordinates (σ, τ) with origin at p in which the metric takes the form*

$$ds^2 = \alpha(\sigma, \tau)(-d\tau^2 + d\sigma^2) \tag{103}$$

Proof. Let p be any point on the surface, let (x, y) be an arbitrary local coordinate system centered at p. In these coordinates the metric takes the general form

$$ds^2 = E(x, y)dx^2 + F(x, y)(dxdy + dydx) + G(x, y)dy^2, \quad EG - F^2 < 0$$

where E,F and G are $\mathcal{C}^2$-functions. Note that the condition $EG - F^2 < 0$ is a consequence of the signature (1,1). Now we define the one-forms

$$\begin{aligned}\theta &= \frac{E}{F + \sqrt{F^2 - EG}} dx + dy \\ \phi &= \left(F + \sqrt{F^2 - EG}\right) dx + G dy\end{aligned}$$

which allows us to write a new expression for the original metric

$$ds^2 = \frac{1}{2}(\theta\phi + \phi\theta)$$

Since θ and ϕ vanish nowhere, by Lemma 1 there locally exist $\mathcal{C}^2$-functions λ, μ, u and v so that $\theta = \lambda du$ and $\phi = \mu dv$. The metric now reads

$$ds^2 = \frac{\lambda\mu}{2}(du dv + dv du)$$

By defining new coordinates

$$\sigma = \frac{u+v}{2}, \quad \tau = \frac{u-v}{2}$$

and setting $\alpha(\sigma,\tau) = \lambda(\sigma,\tau)\mu(\sigma,\tau)$ the metric finally takes the form (103). □

The above proposition even holds for surfaces with arbitrary signatures, but the proof of these cases is much more difficult (see [7] for example).

A.2 Proofs of other Propositions

Lemma 2 *If a metric of the form*

$$ds^2 = \tilde{g}_{ij}(x^0,x^1)dx^i dx^j + F^2(x^0,x^1)d\Omega^2, \quad (i,j) \in \{0,1\} \tag{104}$$

satisfies the following two equations

$$F\tilde{\Delta}F + \tilde{\nabla}_i F \tilde{\nabla}^i F = 1 \tag{105}$$

$$\left(\tilde{\nabla}_i\tilde{\nabla}_j - \frac{1}{2}\tilde{g}_{ij}\tilde{\Delta}\right)F = 0 \tag{106}$$

then it also satisfies

$$\frac{1}{F}\tilde{\nabla}F = \frac{1}{2}\tilde{R} \tag{107}$$

Proof. Choose any point p on the manifold. We first prove the lemma in the case, that not all partial derivatives of F vanish at p, and afterwards we consider the other case.

In the first case, choose $j \in \{0, 1\}$ such that $\tilde{\nabla}_j F(p) \neq 0$. Applying $\tilde{\nabla}_j$ to equation (105) we get

$$\underbrace{\tilde{\nabla}_j F \tilde{\Delta} F}_{A} + \underbrace{F \tilde{\nabla}_j \tilde{\Delta} F}_{B} + \underbrace{\tilde{\nabla}_j \tilde{\nabla}_i F \tilde{\nabla}^i F}_{C} + \underbrace{\tilde{\nabla}_i F \tilde{\nabla}_j \tilde{\nabla}^i F}_{D} = 0 \tag{108}$$

The terms C and D can be rearranged using equation (106):

$$\tilde{\nabla}_j \tilde{\nabla}_i F \tilde{\nabla}^i F = \frac{1}{2} \tilde{g}_{ij} \tilde{\Delta} F \tilde{\nabla}^i F = \frac{1}{2} \tilde{\nabla}_j F \tilde{\Delta} F$$
$$\tilde{\nabla}_i F \tilde{\nabla}_j \tilde{\nabla}^i F = \tilde{\nabla}_i F \tilde{g}^{ik} \frac{1}{2} \tilde{g}_{jk} \tilde{\Delta} F = \frac{1}{2} \tilde{\nabla}_j F \tilde{\Delta} F$$

The term B requires more work. First we need the relations $[\nabla_i, \nabla_j] V = 0$ and $[\nabla_i, \nabla_j] W_k = -R^l_{kij} W_l$ where R is the Riemann tensor, V is any scalar function and W is any one-form. These relations are easily proved using direct calculation. We proceed as follows:

$$\begin{aligned} F \tilde{\nabla}_j \tilde{\Delta} F &= F \tilde{g}^{ik} \tilde{\nabla}_j \tilde{\nabla}_i \tilde{\nabla}_k F = F \tilde{g}^{ik} (\tilde{\nabla}_i \tilde{\nabla}_j \tilde{\nabla}_k F - \tilde{R}^l_{kji} \tilde{\nabla}_l F) \\ &= F \tilde{g}^{ik} (\tilde{\nabla}_i \tilde{\nabla}_k \tilde{\nabla}_j F - \tilde{g}^{ml} \tilde{R}_{lkji} \tilde{\nabla}_m F) \end{aligned}$$

Next we use equation (106), the symmetries of the Riemann tensor $R_{lkji} = -R_{klji} = R_{klij}$ and the definition of the Ricci tensor $R_{lj} = R^k_{lkj}$:

$$F \tilde{\nabla}_j \tilde{\Delta} F = \frac{1}{2} F \tilde{\nabla}_j \tilde{\Delta} F - F \tilde{\nabla}_m F \tilde{g}^{ml} \tilde{R}_{lj}$$

By Proposition 1 we have $\tilde{G}_{lj} = 0 \Rightarrow \tilde{R}_{lj} = \frac{1}{2} \tilde{g}_{lj} \tilde{R}$ so we get

$$\begin{aligned} F \tilde{\nabla}_j \tilde{\Delta} F &= \frac{1}{2} F \tilde{\nabla}_j \tilde{\Delta} F - F \tilde{\nabla}_m F \tilde{g}^{ml} \tilde{\frac{1}{2}} \tilde{g}_{lj} \tilde{R} \\ &= \frac{1}{2} F \tilde{\nabla}_j \tilde{\Delta} F - \frac{1}{2} F \tilde{\nabla}_j F \tilde{R} \\ &\Rightarrow F \tilde{\nabla}_j \tilde{\Delta} F = -F \tilde{\nabla}_j F \tilde{R} \end{aligned}$$

Combining all results, equation (108) reads

$$2 \tilde{\nabla}_j F \tilde{\Delta} F - F \tilde{\nabla}_j F \tilde{R} = 0 \tag{109}$$

At p we have $F \tilde{\nabla}_j F \neq 0$ and this also holds in some neighbourhood of p by continuity. So locally we can divide equation (109) by $2F \tilde{\nabla}_j F$, leading to equation (107), which is the desired result.

In the second case, we assume that all partial derivatives of F vanish at p: $\tilde{\nabla}_0 F = \tilde{\nabla}_1 F = 0$. Now we assume, that there exists a neighbourhood of p, where all derivatives vanish identically and we show that this leads to a contradiction. If the assumption is true, then all second partial derivatives vanish at p and so we have

$$\begin{aligned}(\tilde{\Delta} F)(p) &= (\tilde{g}^{ij}\tilde{\nabla}_i\tilde{\nabla}_j F)(p) \\ &= (\tilde{g}^{ij}\partial_i\partial_j F)(p) - (\tilde{g}^{ij}\tilde{\Gamma}^k{}_{ij}\tilde{\nabla}_k F)(p) = 0 \\ &\Rightarrow (F\tilde{\Delta}F + \tilde{\nabla}_i F\tilde{\nabla}^i F)(p) = 0\end{aligned}$$

which contradicts the field equation (105). So the assumption was wrong and it follows, that in any neighbourhood of p there is a point where one of the partial derivatives does not vanish, or in other words, we can find a sequence of points converging to p so that for every point in the sequence equation (107) holds. Then, by continuity, it also must hold at p, which proves the second case. □

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